1. What is the difference between an internal procedure and an external procedure?

2. How are arguments passed to procedures in FORTRAN?

3. By the end of this exercise you should have a module called quadrature that can estimate an integral in a variety of ways.

   - The first requirement is to create a module that contains the trapezoidal function and one of the adaptive procedures, and to modify the main program to use the quadrature module to perform the integration. Verify that both codes work by integrating the function $\cos^2 \pi x$ between 0 and 1. You will have to make myfunc an external function for the executable to build properly.

   - Write a second version of the trapezoid function that avoids computing the functions twice over the same points when the number of points is doubled. Refer to figure 1 for a graphical illustration. Using $H$ and $h$

   \[ S_N = \frac{b - a}{N} \left[ \frac{f(a)}{2} + f(H + a) + f(2H + a) + f(3H + a) + \ldots ight. \\
   \left. + f((N - 1)H + a) + \frac{f(b)}{2} \right] \quad (1) \]

Figure 1: The solid circles refer to a trapezoidal integration with $N$ points and a grid spacing of $\Delta x = (b - a)/N$, while the open circles refer to an estimate with $2N$ points and a grid spacing of $\Delta x = (b - a)/(2N)$. The function is evaluated twice on points were the solid and unfilled circles overlap for the grid spacing for the $N$ and $2N$ estimates, respectively, we have $h = H/2 = (b - a)/(2N)$; The two trapezoidal estimates can now be written as
\[ S_N = \frac{b-a}{N} \left[ \frac{f(a)}{2} + f(2h + a) + f(4h + a) + f(6h + a) + \ldots \right. \\
\left. + f(2N-2)h + a) + \frac{f(b)}{2} \right] \] (2)

\[ S_{2N} = \frac{b-a}{2N} \left[ \frac{f(a)}{2} + f(h + a) + f(2h + a) + f(3h + a) + f(4h + a) + \right. \\
\left. + f(5h + a) + f(6h + a) + \ldots \right. \\
\left. + f(2(N-1)h + a) + f(2N-1)h + a) + \frac{f(b)}{2} \right] \] (3)

Forming the term \( S_{2N} - S_N/2 \) from equations 2 and 3 we have:

\[ S_{2N} = \frac{S_N}{2} + \frac{b-a}{2N} \left[ f(h + a) + f(3h + a) + f(5h + a) + \ldots + \right. \\
\left. f(2(N-1)h + a) \right] \]

(4)

\[ S_{2N} = \frac{S_N}{2} + \frac{b-a}{2N} \sum_{i=1,2,2N} f(ih + a) \] (5)

Equation 5 expresses the new estimate in terms of the old ones without having to evaluate the function multiple times at the same points; the new evaluations are needed only at the half way points, in other words only the odd indeces from \( i = 1, 3, \ldots, 2N - 1 \) are required. Implement a function that uses equations 5 to calculate the improved estimate and incorporate it in the adaptive routine. Test your code by integrating the function \( \cos^4 \pi x \) between 0 and 1 (the answer should be \( 3/8 \)), and use a tolerance of \( 10^{-6} \) starting with a single interval. Monitor the decrease of the error \( |S_N - 3/8| \) as a function of \( N \) and confirm that each doubling decreases the error by a factor of 4.

- The trapezoidal rule is derived from approximating the curve with straight lines between quadrature points. Another integration rule can be derived by fitting the curve with parabolas, and its integral can be easily evaluated by calling the trapezoidal function twice with \( N \) and \( 2N \) points and combining the two estimates as \( S = (4S_{2N} - S_N)/3 \). Call the new function Simpson. How many points does the Simpson quadrature require to achieve the same accuracy, \( 10^{-6} \) as the trapezoidal rule? How fast does the error decrease?

- Modify the recursive trapezoidal integration function presented in class to avoid repeated evaluation of the function at the intermediate points. Run your code and try to integrate \( x^5 \) with a tolerance of \( 10^{-4} \).
Given three points, A, B and P in the plane with coordinates \(\vec{x}_a, \vec{x}_b,\) and \(\vec{x}_p,\) the vector product \(\vec{A}\vec{B} \times \vec{A}\vec{P}\) can be used to determine on which side of the line \(\vec{A}\vec{B}\) lies the point \(P\). The vector product is
\[
\vec{A}\vec{B} \times \vec{A}\vec{P} = \left((x_b - x_a)(y_p - y_a) - (y_b - y_a)(x_p - x_a)\right)\vec{k}
\]  
where \(\vec{k}\) is a unit vector normal to the plane. If the quantity in bracket, call it \(Z\), is positive the point \(P\) is to the left of the line, otherwise it is to its right. The quantity in bracket is proportional to the signed distance of the point \(P\) to the line \(\vec{A}\vec{B}:\)
\[
d = \frac{(x_b - x_a)(y_p - y_a) - (y_b - y_a)(x_p - x_a)}{\sqrt{(x_b - x_a)^2 + (y_a - y_b)^2}} = \frac{Z}{R} \tag{7}
\]

(a) Write a fortran function that receives as input the point coordinates, \(A,B,\) and \(P,\) and that returns the numerator, \(Z,\) of the expression above. The numerator should be zero if \(P\) falls on the line \(A\vec{B}.\)

(b) Try your code with the following inputs

<table>
<thead>
<tr>
<th>(A)</th>
<th>(B)</th>
<th>(P)</th>
<th>(Z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>(3,1)</td>
<td>(2,2)</td>
<td></td>
</tr>
<tr>
<td>(1,1)</td>
<td>(3,1)</td>
<td>(0,2)</td>
<td></td>
</tr>
<tr>
<td>(1,1)</td>
<td>(3,1)</td>
<td>(5,1)</td>
<td></td>
</tr>
<tr>
<td>(3,1)</td>
<td>(1,3)</td>
<td>(1,1)</td>
<td></td>
</tr>
</tbody>
</table>

(c) The result above can be used to decide whether a point \(P\) is inside or outside a triangle. From the sketch in figure 2 it is easy to convince yourself that \(P\) is inside the triangle \(P\) if it is on the same right side for all three segments: \(A\vec{B}, B\vec{C},\) and \(C\vec{A},\) that is if \(Z(P, A\vec{B}), Z(P, B\vec{C})\) and \(Z(P, C\vec{A})\) have all the same sign. Use this fact, and the previously written code to write a function that returns 1 if \(P\) is inside the triangle, \(-1\) if it is outside and 0 if it falls on one of the segments. PS: It is important to process all three segments in a single order, either going around clockwise or counter clockwise.

(d) Try your code on the triangle defined by the following points \(A = (1,1),\) \(B = (3,1)\) and \(C = (1,3)\) and try the following points

<table>
<thead>
<tr>
<th>(P)</th>
<th>(2,0)</th>
<th>(2,2)</th>
<th>(\left(\frac{5}{2}, 2\right))</th>
<th>(\left(\frac{5}{2}, 2\right))</th>
</tr>
</thead>
<tbody>
<tr>
<td>inside</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>