Chapter 7

Numerical Dispersion of Linearized SWE

This chapter is concerned with the impact of FDA and variable staggering on the fidelity of wave propagation in numerical models. We will use the shallow water equations as the model equations on which to compare various approximations. These equations are the simplest for describing wave motions in the ocean and atmosphere, and they are simple enough to be tractable with pencil and paper. By comparing the dispersion relation of the continuous and discrete systems, we can decide which scales of motions are faithfully represented in the model, and which are distorted. Conversely the diagrams produced can be used to decide on the number of points required to resolve specific wavelengths. The two classes of wave motions encountered here are inertia-gravity waves, and Rossby waves. The main reference for the present work is Dukowicz (1995).

The plan is to look at dynamical system of increasing complexity in order to highlight various aspects of the discrete systems. We start by looking at 1D versions of the linearized shallow water equations, and unstaggered and staggered versions of the discrete approximation; in particular we constrain these two approaches for several high order centered difference scheme and show the superiority of the staggered system. Second we look at the impact of including a second spatial dimensional and include rotation but restrict ourselves to second order schemes; the discussion is instead focussed on the various staggering on the dispersive properties. Lastly we look at the dispersive relation for the Rossby waves.

7.1 Linearized SWE in 1D

Since we are interested in applying Fourier analysis to study wave propagations, we need to linearize the equations and hold the coefficients of the PDE to be constant. For the shallow water equations, they are:

\[
\begin{align*}
  u_t + g \eta_x &= 0 \\
  \eta_t + H u_x &= 0
\end{align*}
\]  

(7.1)  

(7.2)
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7.1.1 Centered FDA on A-grid

The straightforward approach to discretizing the shallow water equation in space is to replace the continuous partial derivatives by their discrete counterparts. The main question is what impact do the choice of variable staggering have on the dispersion relationship. We start by looking at the case where \( u \) and \( \eta \) are co-located. We also restrict ourselves at centered approximation to the spatial derivatives which have the form:

\[
\frac{u_j}{\Delta x} = \sum_{m=1}^{M} \alpha_m (u_{j+m} - u_{j-m}) + O(\Delta x^{2M}) \tag{7.3}
\]

where \( M \) is the width of the stencil; the \( \alpha_m \) are coefficients that can be obtained from the Taylor series expansions (see equations 3.24, 3.27, and 3.29. The order of the neglected term on an equally spaced grid is \( O(\Delta x) \)).

A similar representation holds for the \( \eta \) derivative. The semi-discrete form of the equation is then:

\[
\begin{align*}
\frac{u_t}{\Delta x} + g \sum_{m=1}^{M} \alpha_m (\eta_{j+m} - \eta_{j-m}) &= 0 \tag{7.4} \\
\frac{\eta_t}{\Delta x} + H \sum_{m=1}^{M} \alpha_m (u_{j+m} - u_{j-m}) &= 0 \tag{7.5}
\end{align*}
\]

To compute the numerical dispersion associated with the spatially discrete system, we need to look at periodic solution in space and time, and thus we set

\[
\begin{pmatrix}
  u_j \\
  \eta_j 
\end{pmatrix} =
\begin{pmatrix}
  \hat{u} \\
  \hat{\eta}
\end{pmatrix} e^{i(kx_j - \sigma t)} \tag{7.6}
\]

Hence we have for the time derivative \( u_t = -\sigma \hat{u} e^{i(kx_j - \sigma t)} \), and for the spatially discrete derivative

\[
\begin{align*}
u_{j+m} - u_{j-m} &= \hat{u} \left[ e^{i(kx_{j+m} - \sigma t)} - e^{i(kx_{j-m} - \sigma t)} \right] = \hat{u} e^{i(kx_j - \sigma t)} 2i \sin mk \Delta x; \tag{7.7}
\end{align*}
\]

Hence the FDA of the spatial derivative has the following expression

\[
\sum_{m=1}^{M} \alpha_m (u_{j+m} - u_{j-m}) = 2i \hat{u} e^{i(kx_j - \sigma t)} \sum_{m=1}^{M} \alpha_m \sin mk \Delta x \tag{7.8}
\]

Similar expressions can be written for the \( \eta \) variables. Inserting the periodic solutions in 7.5 we get the homogeneous system of equations for the amplitudes \( \hat{u} \) and \( \hat{\eta} \):

\[
\begin{align*}
-i\sigma \hat{u} + gi \left( \sum_{m=1}^{M} \alpha_m \sin mk \Delta x \right) \hat{\eta} &= 0 \tag{7.9} \\
Hi \left( \sum_{m=1}^{M} \alpha_m \sin mk \Delta x \right) \hat{u} - i\sigma \hat{\eta} &= 0 \tag{7.10}
\end{align*}
\]

For non-trivial solution we require the determinant of the system to be equal to zero, a condition that yields the following dispersion relation:

\[
\sigma = \pm c \left( \sum_{m=1}^{M} \alpha_m \sin mk \Delta x \right) \tag{7.11}
\]
where \( c = \sqrt{gH} \) is the gravity wave speed of the continuous system. The phase speed is then

\[
C_{A,M} = c \left( \sum_{m=1}^{M} \alpha_m \frac{\sin mk\Delta x}{k\Delta x} \right) \tag{7.12}
\]

and clearly the departure of the term in bracket from unity determines the FDA phase fidelity of a given order \( M \). We thus have the following relations for schemes of order 2, 4 and 6:

\[
\begin{align*}
C_{A,2} &= c \frac{\sin k\Delta x}{k\Delta x} \tag{7.13} \\
C_{A,4} &= c \frac{8 \sin k\Delta x - \sin 2k\Delta x}{6k\Delta x} \tag{7.14} \\
C_{A,6} &= c \frac{45 \sin k\Delta x - 9 \sin 2k\Delta x + \sin 3k\Delta x}{30k\Delta x} \tag{7.15}
\end{align*}
\]

### 7.1.2 Centered FDA on C-grid

When the variables are staggered on a C-grid, the spatially discrete equations take the following form

\[
u_t|_{j+\frac{1}{2}} + g \frac{\sum_{m=0}^{M} \beta_m (\eta_{j+\frac{1}{2}+1+2m} - \eta_{j+\frac{1}{2}-1+2m})}{\Delta x} = 0 \tag{7.16}
\]

\[
\eta_t|_{j} + H \frac{\sum_{m=0}^{M} \beta_m (u_{j+\frac{1}{2}+1+2m} - u_{j-\frac{1}{2}+1+2m})}{\Delta x} = 0 \tag{7.17}
\]

where \( \beta_m \)'s are the differentiation coefficients on a staggered grid. These can be obtained from applying the expansion in 3.24 to grids of spacing \( (2m+1)\Delta x/2 \) with \( m = 0, 1, 2, \ldots \) to get:

\[
\frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x} = \frac{\partial u}{\partial x}|_j + \frac{(\Delta x/2)^2 \partial^3 u}{3! \partial x^3} + \frac{(\Delta x/2)^4 \partial^5 u}{5! \partial x^5} + \frac{(\Delta x/2)^6 \partial^7 u}{7! \partial x^7} \tag{7.18}
\]

\[
\frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{3\Delta x} = \frac{\partial u}{\partial x}|_j + \frac{(3\Delta x/2)^2 \partial^3 u}{3! \partial x^3} + \frac{(3\Delta x/2)^4 \partial^5 u}{5! \partial x^5} + \frac{(3\Delta x/2)^6 \partial^7 u}{7! \partial x^7} \tag{7.19}
\]

\[
\frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{5\Delta x} = \frac{\partial u}{\partial x}|_j + \frac{(5\Delta x/2)^2 \partial^3 u}{3! \partial x^3} + \frac{(5\Delta x/2)^4 \partial^5 u}{5! \partial x^5} + \frac{(5\Delta x/2)^6 \partial^7 u}{7! \partial x^7} \tag{7.20}
\]

The fourth order approximation can be obtained by combining these expressions to yield:

\[
\frac{9}{8} \frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x} - \frac{1}{8} \frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{3\Delta x} = \frac{\partial u}{\partial x}|_j - 9 \frac{(\Delta x/2)^4 \partial^5 u}{5! \partial x^5} - 90 \frac{(\Delta x/2)^6 \partial^7 u}{7! \partial x^7} \tag{7.21}
\]

\[
\frac{25}{24} \frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x} - \frac{1}{24} \frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{5\Delta x} = \frac{\partial u}{\partial x}|_j - 25 \frac{(\Delta x/2)^4 \partial^5 u}{5! \partial x^5} - 650 \frac{(\Delta x/2)^6 \partial^7 u}{7! \partial x^7} \tag{7.22}
\]

Finally the above two expressions can be combined to yield the sixth-order approximation:

\[
\frac{450}{384} \frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x} - \frac{25}{128} \frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{3\Delta x} + \frac{9}{384} \frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{5\Delta x} = \frac{\partial u}{\partial x}|_j + 150 \frac{(\Delta x/2)^6 \partial^7 u}{7! \partial x^7} \tag{7.23}
\]
Going back to the dispersion relation, we now look for periodic solution of the form:

\[ u_{j+\frac{1}{2}} = \hat{u}e^{i(kx_j+\frac{1}{2}-\sigma t)} \quad \text{and} \quad \eta_j = \hat{\eta}e^{i(kx_j-\sigma t)} \]  

(7.24)

which when replaced in the FDA yields the following

\[ \sum_{m=0}^{M} \beta_m(u_{j+\frac{1}{2}+2m} - u_{j-\frac{1}{2}+2m}) = \hat{u}e^{i(kx_j-\sigma t)} \sum_{m=0}^{M} \beta_m \left[ e^{i\frac{k+2m}{2}\Delta x} - e^{-i\frac{k+2m}{2}\Delta x} \right] \]  

(7.25)

\[ \sum_{m=0}^{M} \beta_m(\eta_{j+\frac{1}{2}+2m} - \eta_{j-\frac{1}{2}+2m}) = \hat{\eta}e^{i(kx_j+\frac{1}{2}-\sigma t)} \sum_{m=0}^{M} \beta_m \left[ e^{i\frac{k+2m}{2}\Delta x} - e^{-i\frac{k+2m}{2}\Delta x} \right] \]  

(7.27)

Inserting these expressions in the FDA for the C-grid we obtain the following equations after eliminating the exponential factors to get the dispersion equations:

\[ -i\sigma \hat{u} + g2i \left( \sum_{m=0}^{M} \beta_m \sin \frac{1+2m}{2}k\Delta x \right) \hat{\eta} = 0 \]  

(7.29)

\[ H2i \left( \sum_{m=1}^{M} \beta_m \sin \frac{1+2m}{2}k\Delta x \right) \hat{u} - i\sigma \hat{\eta} = 0 \]  

(7.30)

The frequency and the phase speed are then given by

\[ \sigma_{C,M} = \pm c \sum_{m=0}^{M} \beta_m \sin \frac{1+2m}{2}k\Delta x \quad \sigma_{C,M} = \pm c \sum_{m=0}^{M} \frac{\beta_m \sin \frac{1+2m}{2}k\Delta x}{k\Delta x/2} \]  

(7.31)

We thus have the following phase speeds for schemes of order 2, 4 and 6

\[ \sigma_{C,2} = c \frac{\sin \frac{k\Delta x}{2}}{k\Delta x/2} \]  

(7.32)

\[ \sigma_{C,4} = c \left[ \frac{27 \sin \frac{k\Delta x}{2} - 3 \sin 3\frac{k\Delta x}{2}}{24k\Delta x/2} \right] \]  

(7.33)

\[ \sigma_{C,6} = c \left[ \frac{125 \sin \frac{k\Delta x}{2} - 125 \sin 3\frac{k\Delta x}{2} + 9 \sin 5\frac{k\Delta x}{2}}{1920k\Delta x/2} \right] \]  

(7.34)

Figure 7.1 compares the shallow water phase speed for the staggered and unstaggered configuration for various order of centered difference approximations. The un-staggered schemes display a familiar pattern: by increasing order the phase speed in the intermediate wavelengths is improved but there is a rapid deterioration for the marginally resolved waves \( k\Delta x \geq 0.6 \). The staggered scheme on the other hand displays a more accurate representation of the phase speed for the entire spectrum. Notice that the second order
Figure 7.1: Phase speed of the spatially discrete linearized shallow water equation. The solid lines show the phase speed for the A-grid configuration for centered schemes of order 2, 4 and 6, while the dashed lines show the phase speed of the staggered configuration for orders 2, 4 and 6.
staggered approximation provides a phase fidelity which is comparable to the fourth order approximation in the intermediate wavelengths $0.2\pi \leq k\Delta x \leq 0.6\pi$ and superior for wavelengths $k\Delta x \geq 0.6$. Finally, and most importantly the unstaggered scheme possess a null mode where $C = 0$ which could manifest itself as a non-propagating $2\Delta x$ spurious mode; the staggered schemes do not have a null mode.

### 7.2 Two-Dimensional SWE

Here we carry out the dispersion relationship for the two-dimensional shallow water equations in the presence of rotation. We shall consider the two cases of flow on an $f$-plane and flow on a $\beta$-plane. We will also consider various grid information that include the Arakawa grids A, B, C, and D.

#### 7.2.1 Inertia gravity waves

The linearized equations are given by

\begin{align}
    u_t - fv + gn_x &= 0 \\
    v_t + fu + gn_y &= 0 \\
    \eta_t + H(u_x + v_y) &= 0
\end{align}

(7.35) (7.36) (7.37)

Assuming periodic solutions in time and space of the form

$$(u, v, \eta) = (\hat{u}, \hat{v}, \hat{\eta})e^{i(kx + ly - \omega t)},$$

where $(k, l)$ are wavenumbers in the $x - y$ directions, we obtain the following eigenvalue problem for the frequency $\sigma$:

$$\begin{vmatrix}
-\sigma & -f & gik \\
    f & -\sigma & gil \\
iHk & iHl & -\sigma
\end{vmatrix} = -i\omega [-\omega^2 + f^2 + c^2(k^2 + l^2)] = 0$$

(7.38)

Here $c = \sqrt{gH}$ is the gravity wave speed. The non-inertial roots can be written in the following form:

$$\sigma^2 = 1 + a^2 \left[(kd)^2 + (ld)^2\right]$$

(7.39)
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where $\sigma = \omega / f$ is a non-dimensional frequency, $d$ is the grid spacing assumed uniform in both directions and $a$ is the ratio of the Rossby number to the grid spacing, i.e. it is the number of points per rossby radius:

$$a = \frac{R_o}{d} = \frac{gH}{fd}.$$  \hfill (7.40)

Although the continuous dispersion does not depend on a grid spacing, it is useful to write it in the above form for comparison with the numerical dispersion relations. The numerical dispersion for the various grids are given by

$$\sigma^2_A = 1 + a^2 \left[ \sin^2 kd + \sin^2 ld \right]$$ \hfill (7.41)

$$\sigma^2_B = 1 + 2a^2 \left[ 1 - \cos kd \cos ld \right]$$ \hfill (7.42)

$$\sigma^2_C = \cos^2 \frac{kd}{2} \cos^2 \frac{ld}{2} + 4a^2 \left[ \sin^2 \frac{kd}{2} + \sin^2 \frac{ld}{2} \right]$$ \hfill (7.43)

$$\sigma^2_D = \cos^2 \frac{kd}{2} \cos^2 \frac{ld}{2} + a^2 \left[ \cos^2 \frac{kd}{2} \sin^2 ld + \sin^2 kd \cos^2 \frac{ld}{2} \right]$$ \hfill (7.44)

7.3 Rossby waves

The Rossby dispersion relation are given by

$$\sigma = -a^2 kd \left\{ 1 + a^2 \left[ (kd)^2 + (ld)^2 \right] \right\}^{-1}$$ \hfill (7.45)

$$\sigma_A = -a^2 \sin kd \cos ld \left\{ 1 + a^2 \left[ \sin^2 kd + \sin^2 ld \right] \right\}^{-1};$$ \hfill (7.46)

$$\sigma_B = -a^2 \sin kd \left\{ 1 + 2a^2 \left[ 1 - \cos kd \cos ld \right] \right\}^{-1}$$ \hfill (7.47)

$$\sigma_C = -a^2 \sin kd \cos^2 \frac{ld}{2} \left\{ \cos^2 \frac{kd}{2} \cos^2 \frac{ld}{2} + 4a^2 \left[ \sin^2 \frac{kd}{2} + \sin^2 \frac{ld}{2} \right] \right\}^{-1}$$ \hfill (7.48)

$$\sigma_D = -a^2 \sin kd \cos^2 \frac{ld}{2} \left\{ 1 + 4a^2 \left[ \sin^2 \frac{kd}{2} + \sin^2 \frac{ld}{2} \right] \right\}^{-1};$$ \hfill (7.49)

where the frequency is now normalized by $\beta d$. The normalized Rossby wave frequencies, and their relative error for the various grids are displayed in figures 7.8-7.12 for various Rossby radius parameters $a$. Since contour plots are hard to read we also supply line-plots for special $l$ values $l = 0$ and $l = k$ in figure 7.13. From these plots one can conclude the following:

1. All grid configurations have a null mode at $k \Delta x = \pi$

2. The C and D grids have a null mode for all zonal wavenumber when $ld = \pi$.

3. for $a \geq 2$ the B, C and D grids perform similarly for the resolved portion of the spectrum $kd \leq 2\pi/5$. 
Figure 7.3: Comparison of the dispersion relation on the Arakawa A, B, C and D grids. The top figure shows the dispersion relation while the bottom one shows the relative error. The parameter $a=8$. 
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Figure 7.4: Same as 7.3 but for a=4.
Figure 7.5: Same as 7.3 but for a=2.
Figure 7.6: Same as 7.3 but for a=1.
Figure 7.7: Same as 7.3 but for $a=1/2$. 
Figure 7.8: Comparison of Rossby wave dispersion for the different Arakawa grids. The top figures show the dispersion while the bottom ones show the relative error. Here $a=8$. 
Figure 7.9: Same as figure 7.8 but for \( a = 4 \).
Figure 7.10: Same as figure 7.8 but for a=2.
Figure 7.11: Same as figure 7.8 but for \( a=1 \).
Figure 7.12: Same as figure 7.8 but for a=1/2.
Figure 7.13: Rossby wave frequency $\sigma$ versus $k \Delta x$ for, from top to bottom $a = 8, 4, 2, 1$ and $1/2$. The left figures show the case $l = 0$ and the right figures the case $l = k$. The black line refers to the continuous case and the colored line to the A (red), B (blue), C (green), and D (magenta) grids.