MPO 662 – Problem Set 3

It is very important that you discuss the numerical results you produce, particularly if they conform with your expectation about what the solution should look like, and if not what may have caused the problems. Plots with no discussion/commentary are not useful and don’t show us whether you have looked critically at your results.

1 Numerical dispersion

1. Carry out a dispersion analysis for the advection equation $u_t + cu_x = 0$ when the spatial derivative is approximated with a fifth-order upwind scheme:

$$u_t + c \frac{-2u_{j-3} + 15u_{j-2} - 60u_{j-1} + 20u_j + 30u_{j+1} - 3u_{j+2}}{60\Delta x} = 0$$

(1)

Plot the dispersion relationship of the real and imaginary part of the frequency as a function of $\Delta x$ and contrast it to the dispersion relationship obtained from a second order centered difference scheme.

2. Carry out a dispersion analysis for the advection equation $u_t + cu_x = 0$ when a 4-th compact difference scheme is used to compute the special derivative:

$$\frac{1}{4} u_x|_{j-1} + u_x|_j + \frac{1}{4} u_x|_{j+1} = \frac{3}{2} \frac{u_{j+1} - u_{j-1}}{2\Delta x}$$

(2)

Show that the numerical phase speed differs from the real one by a term of $(k\Delta x)^4$ when $k\Delta x \ll 1$. You may use the function `taylor` in matlab to figure out the Taylor series.

3. A wave of length 100 m is being represented on a grid with $\Delta x = 20m$ Estimate the relative phase speed error due to the spatial discretization if the scheme used is 2nd, 4th, and 6th order centered-difference, or a 4-th order compact difference scheme. Is there an effect on the amplitude of the modes. Estimate the amount of damping per time step if an upstream scheme of first, third, and fifth order scheme is used assuming $c = 1$. Use the formula in the notes, and the ones derived in the above exercises.

4. What should the grid spacing be for a second-order scheme to reproduce the right phase speed to about 90% accuracy?

2 Programming RK3

Modify your linear advection program to add the following capabilities:

1. a second-order centered scheme for the spatial discretization,

2. a third-order upwind scheme for the spatial discretization,
3. and a third order, strongly stable Runge-Kutta scheme for the time integration. See below for the details.

Apply the new code to the advection problem of homework number 2. Compute the solution using the new time-discretization and the 2 new spatial discretizations. Contrast the solution obtained with the new schemes with those obtained previously at the end of a full circuit. Use 400 and 800 points only in your numerical experiments. Please re-use as much of your previous codes as possible. This should be easy if you exercised disciplined and followed the software design presented earlier.

The TVD-RK3 scheme is a third order Runge-Kutta time-stepping scheme with 3 stages. Its intermediate stages can be seen as successive corrections to \( u^{n+1} \) using a convex combination of weights. Algorithmically the scheme is:

\[
\begin{align*}
  u^{(1)} &= u^n + \Delta t f(u^n) \\
  u^{(2)} &= \frac{3}{4} u^n + \frac{1}{4} u^{(1)} + \frac{1}{4} \Delta t f(u^{(1)}) \\
  u^{n+1} &= \frac{1}{3} u^n + \frac{2}{3} u^{(2)} + \frac{2}{3} \Delta t f(u^{(2)})
\end{align*}
\]