CONSTRAINED-HAMILTONIAN SHALLOW-WATER DYNAMICS ON THE SPHERE *

F. J. BERON-VERA
RSMAS, University of Miami, Florida, USA

Abstract. Salmon's nearly geostrophic model for rotating shallow-water flow is derived in full spherical geometry. The model, which results upon constraining the velocity field to the height field in Hamilton's principle for rotating shallow-water dynamics, constitutes an important prototype of Hamiltonian balanced models. Instead of Salmon's original approach, which consists in taking variations of particle paths at fixed Lagrangian labels and time, Holm's approach is considered here, namely variations are taken on Lagrangian particle labels at fixed Eulerian positions and time. Unlike the classical quasigeostrophic model, Salmon's is found to be sensitive to the differences between geographic and geodesic coordinates. One consequence of this result is that the β plane approximation, which is included in Salmon's original derivation, is not consistent for this class of model.

Key words: Hamilton's principle, shallow water, balance, sphere

1. Introduction

The rotating shallow-water (SW) equations constitute a paradigm for geophysical fluid motions ranging from fast timescale dynamics, associated with inertia-gravity waves, to slow advective-timescale dynamics, associated with nonlinear vortical motions and Rossby waves (cf. Gill, 1982; Pedlosky, 1987). This set of equations constitute the "primitive" equations on which different approximations are usually performed. In this paper I deal with those approximations which involve the introduction of balance relations or constraints that lead to filtering out the fast degrees of freedom. Terms commonly used to denote the resulting models are "balanced," "constrained," or "intermediate;" the latter, in particular, reflects the fact of being at a level which is in between the primitive equations and the equations for geostrophic motion. For an extensive review on the wide variety of balanced models that exists in the literature the reader is referred to Allen et al. (1990a).

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Of particular interest are those balanced models derived by performing approximations directly in Hamilton's principle (HP) for SW dynamics as proposed by Salmon(1983, hereafter referred to as S83). This procedure allows the fundamental symmetry-based conservation laws of the underlying primitive system to be preserved. The approach consists in substituting leading order balance relations and asymptotic expansions into HP before taking variations. In particular, S83's model is derived by constraining the velocity field to the height field in the form of a geostrophic balance relation, i.e. between the pressure gradient and the Coriolis force. This so-called L1 model, however, was shown to produce less accurate solutions to the SW equations than those produced by other non-Hamiltonian intermediate models (Allen et al., 1990a; Allen et al., 1990b; Barth et al., 1990). This is indicative of the known fact that possession of Hamiltonian structure is no guarantee of model's accuracy. Nevertheless, other balance relation choices—potentially more accurate than that considered by S83—are possible (Allen and Holm, 1996; Allen et al., 2002). This fact makes the L1 model an important prototype of constrained Hamiltonian models, and thus motivates the present study.

The L1 nearly geostrophic model, as well as its relatives the extended-geostrophic Hamiltonian models of Allen and Holm (1996) and Allen et al. (2002), have been derived in the Cartesian coordinates of the β plane approximation. Such approximation relies upon expansion of the equations of motion with respect to geographic (e.g. spherical longitude and latitude) coordinates about some fixed point on the surface of the planet, in inverse powers of the (mean) radius of the planet. The expansion is then truncated at first order but retaining only the first order variation of the Coriolis parameter (the so called β term) and neglecting all metric terms, which are of the same order as the β term! Consequently, the β plane approximation is only valid locally and in geodesic coordinates (Phillips, 1973; Verkley, 1990). These coordinate systems are such that all the derivatives of the metric tensor vanishes identically at the origin and thus locally look like Cartesian coordinates. Geographic coordinates are not geodesic in general, except at the equator where coordinate curves are geodesic curves, e.g. great circles in spherical geometry. Consequently, only at the equator the β plane approximation is valid when written in geographic coordinates, but this region is forbidden for the L1 model.

Remarkable is the fact that the quasigeostrophic (QG) model—perhaps the most exploited model of (slow advective-timescale) intermediate dynamics—has the property of being insensitive to differences between geographic and geodesic coordinates, namely the β plane approximation gives the right QG equations (Pedlosky, 1987; Ripa, 1997a, hereafter referred to as R97). Even though the QG system does not fit within the frame of models of the L1 class, i.e. it does not follow from an approximation made in HP for SW motion, it can be derived from HP but for stationary variations of a particularly chosen action (Virasoro, 1981; Holm and Zeitlin, 1998).

The goal of this paper is to derive an L1 model using non-Cartesian geometry in order to make an assessment of the sensitivity of this model to the difference between geographic and geodesic coordinates. I am not aware of a similar development within the Hamiltonian framework except for the works of Shutts (1989) and Verkley (2001). Shutts (1989) derived a modified version of the Hoskins' (1975) semigeostrophic equations, which are another type of intermediate equations that can be derived from the L1 model through a transformation into "geostrophic coordinates" (Salmon, 1985). Verkley (2001), in turn, presented a derivation of an isentropic L1-type model for application to atmospheric flows; the model derived here is based on SW dynamics. Unlike both Shutts' (1989) and Verkley's (2001) derivations, in this paper I use tools from non-Cartesian tensor algebra, which leads to an invariant formulation for the dynamical equations of the L1 model.

The reminder of the paper is organized as follows. In §2 I set up a mathematical model for the Earth's surface that defines the space in which the analysis is carried out. Section 3 includes a derivation of the general equations for a free particle on the smooth surface of the Earth in invariant form. This is done from HP for a general spheroidal Earth in §3.1. The usual spherical approximation is then applied to the resulting motion equations, which, in particular, are written in geographic coordinates (§3.2). Section 3.3 presents a discussion of the consistency of the so-called planar approximations, which include the classical f and β. Section 4 is devoted to extending into non-Cartesian geometry Holm's (1996, hereafter referred to as H96) general HP for variations of Lagrangian particle labels at fixed Eulerian positions and time. The SW and L1 model equations are derived in §§4.4 and 4.5, respectively, using the spherical Earth's model. The equations are written in a coordinate-invariant fashion on the sphere and then particularized to the common geographic coordinate system. Concluding remarks are given in §5. Appendix A presents various relationships involved in the derivation of the equations. Appendix B is reserved for the discussion and comparison of alternative HPs.

2. Earth's Shape Model

I consider here some basic geophysical facts that relate to the shape of the Earth and the forces acting on its equilibrium surface (e.g. Stommel and Moore, 1989; Ripa, 1995; Ripa, 1997b; R97). The mathematical framework on which the invariant formulation of the equations derived in this paper is based involve concepts from non-Cartesian tensor algebra (e.g. Abraham et al., 1988; Dubrovin et al., 1992) that I start by reviewing first.

2.1. NON-CARTESIAN TENSOR ALGEBRA BACKGROUND

Let S be a two-dimensional manifold, coordinatized by $\mathbf{x} := (x^1, x^2)$. Two-dimensional intrinsic vectors on S at any point $\mathbf{x}$ define the tangent space, $T_{\mathbf{x}}S$. The disjoint union of tangent spaces constitute the tangent bundle, TS. Let $\{e_i\}$ be a basis for $T_{\mathbf{x}}S$ and $\{e^i\}$ for the dual space, $(T_{\mathbf{x}}S)^*$, namely

$$e^i(e_j) = \delta^i_j, \tag{2.1}$$

where δ^i_j are the Kroenecker symbols which equal 1 if $i = j$ and 0 otherwise. Let $T^m_n(T_{\mathbf{x}}S)$ be the space of m-contravariant and n-covariant real valued tensors or, simply, (m,n)-tensors. Vectors $a \in T^1_0(T_{\mathbf{x}}S) = T_{\mathbf{x}}S$ are expressed as $a = a^i e_i$ and covectors $\alpha \in T^0_1(T_{\mathbf{x}}S) = (T_{\mathbf{x}}S)^*$ as $\alpha = \alpha_i e^i$; the quantities $a^i = a(e^i)$ and $\alpha_i = \alpha(e_i)$ are the components of a and α, respectively. (N.B. The convention of summation over repeated lower and upper indices is understood.) In general, a (m,n)-tensor A expresses as

$$A = A^{i_1 \cdots i_m}_{j_1 \cdots j_n} e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes e^{j_1} \otimes \cdots e^{j_n}, \tag{2.2}$$

where $A^{i_1 \cdots i_m}_{j_1 \cdots j_n} = A(e^{i_1}, \cdots, e^{i_m}, e_{j_1}, \cdots, e_{j_n})$ and $\otimes$ denotes the tensor product.

Assume now that S is endowed with a Riemannian metric, namely a symmetric, positive definite, bilinear form

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle := m_{ij}\, e^i \otimes e^j, \tag{2.3}$$

where $m_{ij}(\mathbf{x}) := \langle\!\langle e_i, e_j \rangle\!\rangle$. The inner product of two vectors $a, b \in T_{\mathbf{x}}S$ is computed with respect to the metric, i.e.

$$\langle\!\langle a, b \rangle\!\rangle = \left(m_{ij}\, e^i \otimes e^j\right) \cdot (a, b) = m_{ij}\, e^i(a)\, e^j(b) = m_{ij}\, a^i b^j. \tag{2.4}$$

In particular, the square of the distance between two nearby positions on S, $\mathbf{x}$ and $\mathbf{x} + \mathrm{d}\mathbf{x}$, is given by

$$\mathrm{d}s^2 = \langle\!\langle \mathrm{d}\mathbf{x}, \mathrm{d}\mathbf{x} \rangle\!\rangle = \|\mathrm{d}\mathbf{x}\|^2 = m_{ij}\, \mathrm{d}x^i \mathrm{d}x^j. \tag{2.5}$$

Let $^\flat$ be the index lowering operator, and $^\natural$, its inverse, be the index raising operator, which are defined by

$$^\flat : T_{\mathbf{x}}S \to (T_{\mathbf{x}}S)^*;\ a \mapsto \langle\!\langle a, \cdot \rangle\!\rangle \quad \text{and} \quad ^\natural : (T_{\mathbf{x}}S)^* \to T_{\mathbf{x}}S, \tag{2.6}$$

respectively. The matrix of $^\flat$ is $[m_{ij}]$, i.e. $(a^\flat)_i = m_{ij}\, a^j =: a_i$, whereas that of $^\natural$ is $[m_{ij}]^{-1} = m^{-1}\,\mathrm{adj}[m_{ij}] =: [m^{ij}]$, i.e. $(\alpha^\natural)^i = m^{ij}\alpha_j =: \alpha^i$. Here, $m := \det[m_{ij}]$ and adj denotes adjoint (transpose cofactor).

Let, in addition, reserve the symbol $\mathbf{d}$ to denote the exterior derivative (or generalized gradient operator), whose action on a skew-symmetric $(0,k)$-tensor or k-form α, i.e.

$$\alpha = \alpha_{i_1\cdots i_k}\, e^{i_1} \wedge \cdots \wedge e^{i_k}, \tag{2.7}$$

where $\wedge$ denotes the exterior product, is defined by

$$\mathbf{d}\alpha := \sum_{j,i_1<\cdots<i_k} \partial_j \alpha_{i_1\cdots i_k}\, e^j \wedge e^{i_1} \wedge \cdots \wedge e^{i_k}. \tag{2.8}$$

Notice that, in particular, if $k = 0$ then α is simply a scalar and, hence, $\mathbf{d}\alpha = \alpha_{,i} e^i =: \operatorname{grad}\alpha$. N.B. The shorthand notations $\partial_i(\cdot)$ and $(\cdot)_{,i}$ for partial differentiation $\partial(\cdot)/\partial x^i$ are in use.

Finally, let $\mathbb{P}$ be a linear map, with matrix elements $\mathbb{P}_{ij} = \sqrt{m_{ij}}$ for $i = j$ and $\mathbb{P}_{ij} = 0$ otherwise. Then $\mathbb{P} \cdot a$ (resp., $\mathbb{P}^{-1} \cdot a^\flat$) denotes the physical—nontensorial—contravariant (resp., covariant) counterpart of vector a. For orthogonal coordinates, i.e. with $m_{ij} = 0$ for $i \neq j$, physical contravariant and covariant counterparts coincide, namely $\mathbb{P} \cdot a \equiv \mathbb{P}^{-1} \cdot a^\flat$.

2.2. GENERAL ASSUMPTIONS ON S

Two main assumptions make the two-dimensional manifold S an idealized model of the surface of the (solid) Earth. First, S is assumed to be embedded in a three-dimensional Euclidean space which rotates steadily, with spinning frequency Ω, with respect to a Newtonian inertial space. Second, S is assumed to be a geopotential surface. Namely the projections onto $T_{\mathbf{x}}S$ of the ***centrifugal force*** (due to the spinning of the planet with respect to an inertial reference frame) and the ***gravitational attraction*** (due to the deviation of the shape of the planet from a perfect sphere and to inhomogeinities in the mass distribution within the planet) are assumed to balance one another exactly.

As a consequence of the second assumption it follows that

$$\Phi := V + V_{\mathrm{C}} = \text{const.} \tag{2.9}$$

on S, where V is the ***gravitational potential***, V_{C} stands for the ***centrifugal potential***, and their sum, Φ, defines the ***geopotential***. (The constant in the above expression is arbitrary and can be freely set to zero.) The centrifugal potential (per unit mass) can be expressed in invariant form as

$$V_{\mathrm{C}} = -\tfrac{1}{2}\, \|\sigma\|^2, \tag{2.10}$$

where σ is the velocity of the $\mathbf{x}$-system with respect to a suitable inertial frame.

Finally, the ***acceleration of gravity*** is defined as the minus gradient of Φ, thereby determining the vertical direction at each point $\mathbf{x}$ on S. Its magnitude is thus given by

$$g(\mathbf{x}) = \|\mathrm{grad}\,\Phi\| = \sqrt{m^{ij}\Phi_{,i}\Phi_{,j}}. \tag{2.11}$$

2.3. SPHERICAL MODEL

It is convenient—and quite accurate—to consider S as a (two-dimensional) sphere of radius R, say, but keeping the main effect of the gravitational force. Namely that it can sustain a steady rotation, relative to an inertial frame, in any point on S. Thus let the coordinates on S be given by

$$x^1 = (\lambda - \lambda_0)\, R \cos\vartheta_0, \quad x^2 = (\vartheta - \vartheta_0)\, R, \tag{2.12}$$

which are rescaled longitude, λ, and latitude, ϑ, that will be referred here to as ***geographic coordinates***. In this case one can introduce the usual notations (x, y) for (x^1, x^2) and $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ for (e^1, e^2). The corresponding metric matrix, velocity of the $\mathbf{x}$-system, and centrifugal potential, respectively, read:

$$[m_{ij}] = \begin{bmatrix} \gamma^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma = \frac{f}{2\tau\gamma}\hat{\mathbf{x}}, \quad V_{\mathrm{C}} = -\frac{f^2}{8\tau^2}. \tag{2.13}$$

Here,

$$f := 2\Omega \sin\vartheta, \quad \gamma := \sec\vartheta_0 \cos\vartheta, \quad \tau := R^{-1}\tan\vartheta; \tag{2.14}$$

the first parameter is the Coriolis parameter whereas the other two are the geometric coefficients as defined by Ripa [2000a,b]. Consistently with this spherical approximation, the acceleration of gravity is taken as a constant, namely $g \approx 9.8\ \mathrm{m}^2\,\mathrm{s}^{-1}$.

More accurate models (not treated here) should account explicitly for the flattening of the planet at the poles. For instance, although still crude, next in accuracy can be mentioned one that has the form of an axisymmetric spheroid of revolution (Chandrasekhar, 1969; cf. also R97).

3. Particle Dynamics

In this section the manifold S is assumed to represent a smooth and frictionless Earth's surface on which a particle moves freely. The derivation of the particle's equations of motion is instructive inasmuch as it sets the grounds for tackling the more complicated problem of the following section. In particular, it shows clearly how the Coriolis force—which finds its origin in the gravitational force—arises directly from a HP with an action appropriate for an inertial observer, but written in coordinates fixed to the planet. The method is in essence the same as the one used by Pierre Simon de Laplace (1749–1827) to introduce this force over quarter a century before than Gaspard Gustave de Coriolis (1792–1843) was born (cf. R95; R96). The analysis of the particle's equations allows, in addition, one to simplify the discussion on the consistency of the so-called planar approximations (cf. R97).

3.1. GENERAL EQUATIONS

From an inertial observer viewpoint, the only force acting on the particle is the gravitational one. The particle's kinetic and potential energies (per unit mass) as measured by this observer are given by

$$T(\mathbf{x}, \dot{\mathbf{x}}) := \tfrac{1}{2} \left\| \dot{\mathbf{x}} + \sigma \right\|^2, \quad V(\mathbf{x}) = -V_{\mathrm{C}} = \tfrac{1}{2} \left\| \sigma \right\|^2, \tag{3.1}$$

respectively, where the overdot denotes time differentiation and a zero value of the geopotential has been assigned to the Earth's surface. The Lagrangian function, $L : TS \to \mathbb{R}$, is constructed in the usual way, i.e.

$$L(\mathbf{x}, \dot{\mathbf{x}}) := T - V = \tfrac{1}{2} \left\| \dot{\mathbf{x}} \right\|^2 + \langle\langle \dot{\mathbf{x}}, \sigma \rangle\rangle. \tag{3.2}$$

Let δt be a time displacement and $\delta \mathbf{x} := \mathrm{d}/\mathrm{d}\varepsilon|_{\varepsilon=0}\, \mathbf{x}(t + \varepsilon \delta t)$ a variation of the curve $\mathbf{x} : [t_0, t_1] \to T_{\mathbf{x}} S$. Let, in addition,

$$\mathcal{S}[\mathbf{x}] := \int_{t_0}^{t_1} \mathrm{d}t\, L : \mathcal{F}([t_0, t_1]) \to \mathbb{R} \tag{3.3}$$

be the action functional, where $\mathcal{F}([t_0, t_1])$ denotes the set of sufficiently smooth real valued functions on $[t_0, t_1]$. Subject to fixed endpoint conditions, i.e. $\delta \mathbf{x}(t_0) = 0 = \delta \mathbf{x}(t_1)$, the first variation of $\mathcal{S}$, defined as $\delta \mathcal{S} := \mathrm{d}/\mathrm{d}\varepsilon|_{\varepsilon=0}\, \mathcal{S}[\mathbf{x} + \varepsilon \delta \mathbf{x}]$, is given by

$$\begin{aligned} \delta \mathcal{S} &= \int_{t_0}^{t_1} \mathrm{d}t \, \left(L_{,i} \delta x^i + \frac{\partial L}{\partial \dot{x}^i} \delta \dot{x}^i \right) \\ &= \int_{t_0}^{t_1} \mathrm{d}t \, \left(L_{,i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}^i} \right) \delta x^i \\ &= \int_{t_0}^{t_1} \mathrm{d}t \, \left[\tfrac{1}{2} \left(m_{jk,i} - m_{ki,j} - m_{ji,k} \right) \dot{x}^j \dot{x}^k - \ddot{x}_i - \left(\sigma_{i,j} - \sigma_{j,i} \right) \dot{x}^j \right] \delta x^i . \end{aligned} \tag{3.4}$$

HP ($\delta \mathcal{S} = 0$) then yields the Newton's law for the particle in covariant form

$$\boxed{\mathrm{D} \dot{\mathbf{x}}^\flat / \mathrm{d}t + \mathbf{d} \sigma^\flat \cdot \dot{\mathbf{x}} = 0.} \tag{3.5}$$

In this equation, the coordinate representation of the object $\mathrm{D}\alpha/\mathrm{d}t$, for any covector α, is given by $(\mathrm{D}\alpha/\mathrm{d}t)_i = \dot{\alpha}_i - \Gamma^k_{ij}\dot{x}^j\alpha_k$, where $\Gamma^k_{ij}(\mathbf{x}) := \frac{1}{2}m^{kl}(m_{il,j}+m_{jl,i}-m_{ij,l})$ are the Christoffel symbols (of second kind), which establish the (Levi–Civita) connection on the Riemannian manifold S. In addition,

$$\mathbf{d}\sigma^\flat = \sigma_{i,j}\, e^i \wedge e^j = (\sigma_{i,j} - \sigma_{j,i})\; e^i \otimes e^j, \tag{3.6}$$

which can be regarded as the ***Coriolis two-form***. Notice that $\mathbf{d}\sigma^\flat \cdot \dot{\mathbf{x}} = [(\sigma_{i,j} - \sigma_{j,i})\; e^i \otimes e^j] \cdot \dot{\mathbf{x}} = (\sigma_{i,j} - \sigma_{j,i})\; e^i(\cdot)\, e^j(\dot{\mathbf{x}}) = (\sigma_{i,j} - \sigma_{j,i})\, \dot{x}^j\, e^i$. The operator $^\natural$ transforms (3.5) into its contravariant counterpart

$$\mathrm{D}\dot{\mathbf{x}}/\mathrm{d}t + (\mathbf{d}\sigma^\flat \cdot \dot{\mathbf{x}})^\natural = 0; \tag{3.7}$$

here, $\mathrm{D}a/\mathrm{d}t$, for any covector a, in components reads $(\mathrm{D}a/\mathrm{d}t)^i = \dot{a}^i + \Gamma^i_{jk}\dot{x}^j a^k$.

Equations (3.5) or (3.7) are invariant under general coordinate transformations on S, which in this case is not restricted to the spherical Earth model. In particular, these equations nicely show that the Coriolis term is responsible for the particle's trajectory to depart from a geodesic curve on S, i.e. a pure Galilean inertial motion. The latter is only consistent with motions with sufficiently large initial kinetic energy as shown by R97, who described all possible solutions on a sphere, namely the so-called inertial oscillations.

3.2. EQUATIONS ON THE SPHERE IN GEOGRAPHIC COORDINATES

In the geographic coordinate system (2.12) of the spherical Earth's model the only nonzero Christoffel symbols are $\Gamma^2_{11} = \gamma^2\tau$ and $\Gamma^1_{12} = \Gamma^1_{21} = -\tau$. In turn, the matrix of the Coriolis two-form takes the form

$$[\sigma_{i,j} - \sigma_{j,i}] = \begin{bmatrix} 0 & -\gamma f \\ \gamma f & 0 \end{bmatrix}. \tag{3.8}$$

Thus equations (3.5), with the spherical approximation and particularized to geographic coordinates, take the following component representation:

$$\left.\begin{aligned} \ddot{x}_1 - \gamma^2\tau\dot{x}^1\dot{x}_2 - (\gamma f + \tau\dot{x}_1)\dot{x}^2 = 0,\\ \ddot{x}_2 + (\gamma f + \tau\dot{x}_1)\dot{x}^1 = 0. \end{aligned}\right\} \tag{3.9}$$

Let $\mathbf{u} := (u, v) := \mathbb{P} \cdot \dot{\mathbf{x}}$ ($\equiv \mathbb{P}^{-1} \cdot \dot{\mathbf{x}}^\flat$ since the coordinates are orthogonal; cf. §2.1). Application of $\mathbb{P}^{-1}$ transforms set (3.9) into the more familiar form (e.g. R97)

$$\left.\begin{aligned} \dot{u} - (f + \tau u)v = 0,\\ \dot{v} + (f + \tau u)u = 0, \end{aligned}\right\} \tag{3.10}$$

which can be written in vector notation as well, i.e.

$$\dot{\mathbf{u}} + (f + \tau u)\,\hat{\mathbf{z}} \times \mathbf{u} = 0, \tag{3.11}$$

where $\hat{\mathbf{z}}$ is the vertical unit vector and $\times$ denotes the cross product of vectors.

3.3. "PLANAR" APPROXIMATIONS

In addition to the spherical approximation, other standard approximations introduced in the equations are the "planar" approximations. These approximations, which are meant to be valid locally at a point on the sphere in geographic coordinates, are obtained by expanding the equations in inverse powers of the radius of the sphere R. The most common approximations being the f and β. The former is a consistent zeroth-order approximation. The latter, however, is an inconsistent first-order approximation, except at the equator. A consistent nth-order approximation is understood as one that produces $O(R^{-n-1})$ errors in the integrals of motion associated with the equations on the sphere. These integrals are the (kinetic) energy of the particle as measured by a terrestrial observer,

$$E := \tfrac{1}{2}\mathbf{u}^2, \tag{3.12}$$

and the absolute angular momentum (with respect to the center of the planet and in the direction of the axis of rotation), which, up to some constants, is given by

$$M := \gamma u - \Omega R\,(\cos\vartheta_0 - \gamma\cos\vartheta)\,. \tag{3.13}$$

R97 showed that a consistent first-order "planar" approximation must have

$$\textbf{\textit{Ripa “plane”}} : \gamma = 1 - \tau_0 y,\ \tau = \tau_0/\gamma,\ f = f_0 + \beta y/\gamma \tag{3.14}$$

where $\tau_0 := R^{-1}\tan\vartheta_0$, $f_0 := 2\Omega\sin\vartheta_0$, and $\beta := 2\Omega R^{-1}\cos\vartheta_0$. With this approximation the equations of motion conserve $\frac{1}{2}\dot{\mathbf{x}}^2 - \tau_0 y \dot{x}^2 = E - O(R^{-2})$ and $(1-\tau_0 y)u - f_0 y - \frac{1}{2}\beta(1-R^2\tau_0^2)y^2 = M - O(R^{-2})$. The f plane approximation has

$$\textbf{\textit{f plane}} : \gamma = 1,\ \tau = 0,\ f = f_0, \tag{3.15}$$

which consistently implies conservation of $\frac{1}{2}\dot{\mathbf{x}}^2 = E - O(R^{-1})$ and $u - f_0 y = M - O(R^{-1})$. The β plane approximation, in turn, has

$$\boldsymbol{\beta}\ \textbf{\textit{plane}} : \gamma = 1,\ \tau = 0,\ f = f_0 + \beta y \tag{3.16}$$

and implies conservation of $\frac{1}{2}\dot{\mathbf{x}}^2$ and $u - f_0 y - \frac{1}{2}\beta y^2$, which produce $O(R^{-1})$ errors to E and M, respectively, everywhere except at $\vartheta_0 = 0$ where these errors are $O(R^{-2})$ because $\tau_0 \equiv 0$.

It is thus clear that a consistent first-order approximation must include, in general, non-Cartesian terms in order to correctly reproduce the conservation laws of the system. (That is the reason for the quotation marks in this section.) It is worthwhile remarking that this is no longer necessary for motions around the equator. Geographic coordinates at the equator are ***geodesic coordinates*** because all the derivatives of the metric vanish there. For this reason locally at the equator the geometry in geographic coordinates looks like Cartesian and, hence, the β plane is a consistent approximation there. In general, for any point of a space with a symmetric affine connection coordinatized by x^i, $i = 1, 2, \cdots$, say, there exists a coordinate system x'^i, $i = 1, 2, \cdots$, say, such that the coefficients of the connection vanish identically. Such a system can be defined implicitly by $x^i = x'^i - \frac{1}{2}\Gamma^i_{jk}(0)\, x'^j x'^k$ which can be readily seen to result in $\Gamma'^i_{jk}(0) \equiv 0$. For geographic coordinates the transformation $(x', y') \mapsto (x, y)$ reads $(x, y) = (x', y') + \tau_0 x'(y', -\frac{1}{2}x')$, which reduces to the identity at $\vartheta = \vartheta_0$. Of course, the practical use of geodesic coordinates (away from the equator) is questionable (cf. Phillips, 1973; Verkley, 1990).

4. Fluid Dynamics

In this section I derive from HP the equations of motion for (inviscid, unforced) SW and L1 dynamics on the spherical model for the Earth's surface. The derivation makes use of H96's approach but extended to non-Cartesian geometry. In this approach variations of Lagrangian particle labels are performed at fixed Eulerian positions and time. One advantage of H96's approach is that the equations result directly in Eulerian coordinates.

4.1. LAGRANGIAN AND EULERIAN COORDINATES

Identification of fluid particles in a SW motion requires two-dimensional labels $\mathfrak{x} := (\mathfrak{x}^1, \mathfrak{x}^2)$, say, which are defined in certain affine (metricless) space $\mathfrak{S}$, say. Let

$$\varphi \times \mathrm{id} : \mathfrak{S} \times \mathbb{R} \to S \times \mathbb{R};\ (\mathfrak{x}, t) \mapsto (\mathbf{x}, t) = (\varphi(\mathfrak{x}, t), t) \tag{4.1}$$

be the map that relates the Lagrangian labels with the Eulerian two-dimensional positions at time t, and consider its inverse:

$$\varphi^{-1} \times \mathrm{id} : S \times \mathbb{R} \to \mathfrak{S} \times \mathbb{R};\ (\mathbf{x}, t) \mapsto (\mathfrak{x}, t) = (\varphi^{-1}(\mathbf{x}, t), t). \tag{4.2}$$

Let now J and $\mathfrak{J}$ be the Jacobians of these maps, respectively, which are defined by

$$J := \det[J^i_{\mathfrak{i}}], \quad J^i_{\mathfrak{i}} := \partial x^i / \partial \mathfrak{x}^{\mathfrak{i}}, \tag{4.3a}$$

$$\mathfrak{J} := \det[\mathfrak{J}^{\mathfrak{i}}_i], \quad \mathfrak{J}^{\mathfrak{i}}_i := \partial \mathfrak{x}^{\mathfrak{i}} / \partial x^i. \tag{4.3b}$$

The time derivative of a Lagrangian label, following a fluid particle, is zero by construction. Consequently, $\dot{\mathbf{x}} = \partial_t \varphi + \varphi_{,\mathfrak{i}} \dot{\mathfrak{x}}^{\mathfrak{i}} \equiv \partial_t \varphi$. The latter defines the ***Lagrangian or material velocity***

$$\mathfrak{v}(\mathfrak{x}, t) := \partial_t \varphi; \tag{4.4}$$

the ***Eulerian or spatial velocity***, in turn, is defined by

$$\mathbf{v}(\mathbf{x}, t) := \mathfrak{v}(\mathfrak{x}, t). \tag{4.5}$$

Finally, the time derivative of any scalar function $a(\mathbf{x}, t)$ is $\dot{a} = (\partial_t + v^i \partial_i) a =: \mathrm{D}a/\mathrm{D}t$, where $v^i = \mathbf{v}(e^i)$.

4.2. VOLUME CONSERVATION

Let $R(t) \subset S$ be a material spherical cap (made of the same fluid particles) and let $h(\mathbf{x}, t)$ be the depth of the fluid. Let, in addition, $h_0(\mathfrak{x})$ be the density of Lagrangian labels in container $R(t)$. Since $R(t)$ is material, the Lagrangian labels are defined in certain fixed region $\mathfrak{R} \subset \mathfrak{S}$. As a consequence of the metricless nature of $\mathfrak{S}$, the following equality holds:

$$\int_{R(t)} \mathrm{d}^2\mathbf{x} \sqrt{m} h = \int_{\mathfrak{R}} \mathrm{d}^2\mathfrak{x}\, h_0. \tag{4.6}$$

The latter implies

$$\boxed{\sqrt{m} h J = h_0,} \tag{4.7}$$

which is the Lagrangian form of the volume conservation law. In order to obtain the Eulerian counterpart of this law, one needs to take the time derivative of the l.h.s. of (4.6), i.e.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{R(t)} \mathrm{d}^2\mathbf{x} \sqrt{m} h &= \int_{\mathfrak{R}} \mathrm{d}^2\mathfrak{x} \left[\frac{\mathrm{d}J}{\mathrm{d}t} \sqrt{m} h + J \frac{\mathrm{d}}{\mathrm{d}t} \left(\sqrt{m} h \right) \right] \\ &= \int_{\mathfrak{R}} \mathrm{d}^2\mathfrak{x} \sqrt{m} J \left[\partial_t h + \mathrm{div}(h\mathbf{v}) \right], \end{aligned} \tag{4.8}$$

where the relationships of appendix A have been used. The conservation law follows upon setting to zero the latter result:

$$\boxed{\partial_t h + \mathrm{div}(h\mathbf{v}) = 0,} \tag{4.9}$$

where $\operatorname{div}(h\mathbf{v}) := \partial_i\left(\sqrt{m}hv^i\right)/\sqrt{m}$. Notice that in geographic coordinates $\operatorname{div}(h\mathbf{v}) = \gamma^{-1}[\partial_x(hu) + \partial_y(\gamma hv)] =: \nabla\cdot(h\mathbf{u})$.

4.3. GENERAL HP IN EULERIAN COORDINATES

Following H96, I consider an action functional of the form

$$\mathcal{S}[\mathfrak{r}] := \int_{t_0}^{t_1} \mathrm{d}t\, L[\mathbf{v},\mathfrak{J}] = \int_{t_0}^{t_1} \mathrm{d}t \int_D \mathrm{d}^2\mathbf{x}\, l(\mathbf{v},\mathfrak{J};\mathbf{x}), \tag{4.10}$$

where D is a fixed region on S with solid boundary ∂D. Here, L is the Lagrangian functional and, unlike H96 who adopted Cartesian coordinates, $l/\sqrt{m}$ is the Lagrangian density. Variations of Lagrangian particle labels at fixed Eulerian positions and time result in

$$\begin{aligned}\delta\mathcal{S} &= \int\left(\frac{\delta L}{\delta v^i}\delta v^i + \frac{\delta L}{\delta\mathfrak{J}}\delta\mathfrak{J}\right)\\ &= \int \mathfrak{J}J^i_{\mathrm{i}}\delta\mathfrak{r}^{\mathrm{i}}\left[\frac{\mathrm{D}}{\mathrm{D}t}\left(J\frac{\delta L}{\delta v^i}\right) + J\frac{\delta L}{\delta v^j}\partial_i v^j - \partial_i\frac{\delta L}{\delta\mathfrak{J}}\right]\\ &\quad+\int\partial_i\left(\delta x^i\mathfrak{J}\frac{\delta L}{\delta\mathfrak{J}} - v^iJ^j_{\mathrm{i}}\delta\mathfrak{r}^{\mathrm{i}}\frac{\delta L}{\delta v^j}\right)\\ &\quad-\int\partial_t\left(J^i_{\mathrm{i}}\delta\mathfrak{r}^{\mathrm{i}}\frac{\delta L}{\delta v^i}\right),\end{aligned} \tag{4.11}$$

where $\int(\cdot) := \int_{t_0}^{t_1}\mathrm{d}t\int_D \mathrm{d}^2\mathbf{x}\,(\cdot)$. Derivation of (4.11) involved the use of the relationships of appendix A. Fixed endpoint conditions, $\delta\mathfrak{r}(\mathbf{x},t_1) = 0 = \delta\mathfrak{r}(\mathbf{x},t_2)$, allows one to get rid of the last integral in (4.11). Then HP implies the motion equation

$$\boxed{\partial_t\left(J\frac{\delta L}{\delta\mathbf{v}}\right) + \boldsymbol{\pounds}_{\mathbf{v}}\left(J\frac{\delta L}{\delta\mathbf{v}}\right) - \mathbf{d}\frac{\delta L}{\delta\mathfrak{J}} = 0} \tag{4.12}$$

and the no-flow boundary condition

$$\boxed{\langle\!\langle\mathbf{v},n\rangle\!\rangle = 0 \quad @ \quad \partial D,} \tag{4.13}$$

where n is the external normal to the boundary. In (4.12), $\boldsymbol{\pounds}_a\alpha := \mathbf{d}\alpha\cdot a + \mathbf{d}\langle\!\langle a,\alpha\rangle\!\rangle$ is the Lie derivative of covector α along vector a; in components $(\boldsymbol{\pounds}_a\alpha)_i = a^j\alpha_{i,j} + \alpha_j a^j_{,i}$. Result (4.13), in turn, made use of Gauss' theorem, namely $\int_D \mathrm{d}^2\mathbf{x}\,\sqrt{m}\operatorname{div}a = \int_D \mathrm{d}^2\mathbf{x}\,\partial_i(\sqrt{m}a^i) = \oint_{\partial D}\mathrm{d}s\,a^i n_i$ for all vector a.

Finally, it must be mentioned that the Euler–Poincaré formalism provides an alternative way to obtaining (4.12)–(4.13) (Holm et al., 2002).

4.4. HP FOR SW DYNAMICS ON THE SPHERE

Under the assumption that the layer of fluid is thin enough so that it does not represent a source of gravitation, an appropriate Lagrangian density for a HP for SW dynamics on the sphere has

$$\boxed{l(\mathbf{v},\mathfrak{J};\mathbf{x}) := h_0\mathfrak{J}\left(\tfrac{1}{2}\left\|\mathbf{v}\right\|^2 + \langle\!\langle\mathbf{v},\sigma\rangle\!\rangle\right) - \tfrac{1}{2}g\sqrt{m}\left(\frac{h_0\mathfrak{J}}{\sqrt{m}} - H\right)^2} \tag{4.14}$$

along with the definitions

$$\boxed{h := h_0\mathfrak{J}/\sqrt{m}, \quad p := g\,(h - H)\,.} \tag{4.15}$$

Here, h_0 and g are both constants, and $p(\mathbf{x},t)$ is the hydrostatic pressure, where $H(\mathbf{x})$ is the reference depth including the possibility of an irregular topography. The choice $h_0 = \text{const.}$ is necessary in order for the Lagrangian density to be independent of the Lagrangian labels. The assumption $g = \text{const.}$, in turn, is consistent with the spherical approximation for the Earth's surface. The last term on the r.h.s. of (4.14), which is not present in (3.2), relates to the gravitational potential of the fluid column due to the departure of the free surface from the resting position.

According to

$$\frac{\delta L}{\delta\mathbf{v}} = h_0\mathfrak{J}\,(\mathbf{v}+\sigma)^\flat\,, \tag{4.16}$$

$$\frac{\delta L}{\delta\mathfrak{J}} = h_0\left(\tfrac{1}{2}\left\|\mathbf{v}\right\|^2 + \langle\!\langle\mathbf{v},\sigma\rangle\!\rangle - p\right), \tag{4.17}$$

equations (4.12) imply the following equivalent sets of equations:

$$\boxed{\begin{aligned}&\text{a. } \partial_t\,(\mathbf{v}+\sigma)^\flat + \boldsymbol{\pounds}_{\mathbf{v}}\,(\mathbf{v}+\sigma)^\flat + \mathbf{d}\left(p - \tfrac{1}{2}\left\|\mathbf{v}\right\|^2 - \langle\!\langle\mathbf{v},\sigma\rangle\!\rangle\right) = 0,\\ &\text{b. } \partial_t\mathbf{v}^\flat + \mathbf{d}\,(\mathbf{v}+\sigma)^\flat\cdot\mathbf{v} + \mathbf{d}\left(p + \tfrac{1}{2}\left\|\mathbf{v}\right\|^2\right) = 0,\\ &\text{c. } (\partial_t + \nabla_{\mathbf{v}})\mathbf{v}^\flat + \mathbf{d}\sigma^\flat\cdot\mathbf{v} + \mathbf{d}p = 0.\end{aligned}} \tag{4.18}$$

Equation (4.18b) involves the identity $\boldsymbol{\pounds}_a\alpha = \mathbf{d}\alpha\cdot a + \mathbf{d}\langle\!\langle a,\alpha\rangle\!\rangle$, particularized for $a = \mathbf{v}$ and $\alpha = (\mathbf{v}+\sigma)^\flat$. Equation (4.18c), in turn, $\mathbf{d}\langle\!\langle a,\alpha\rangle\!\rangle = \nabla_a\alpha + \nabla_{\alpha^\natural}a^\flat - \mathbf{d}\alpha\cdot a - \mathbf{d}a^\flat\cdot\alpha^\natural$, specialized for $a = \mathbf{v}$ and $\alpha = \mathbf{v}^\flat$. Here, $\nabla_a\alpha$ denotes the covariant derivative of covector α in the direction of vector a; in components $(\nabla_a\alpha)_i = a^k\alpha_{i,k} - \Gamma^j_{ik}\alpha_j a^k$. Any set selected from (4.18) together with the volume conservation equation (4.9), all subject to the no-flow boundary condition (4.13), constitute the covariant form of the

SW equations on a region D defined on the sphere. These equations (or their contravariant counterpart via the metric) are invariant under general changes of coordinates on the sphere.

The SW system conserves energy and Casimirs, namely

$$\mathcal{E} := \tfrac{1}{2}\int_D \mathrm{d}^2\mathbf{x}\,\sqrt{m}\left(h\,\|\mathbf{v}\|^2 + p^2/g\right), \quad \mathcal{C} := \int_D \mathrm{d}^2\mathbf{x}\,\sqrt{m}hC(q), \tag{4.19}$$

for arbitrary $C(\cdot)$ and where

$$qh := \frac{1}{\sqrt{m}h_0}\varepsilon^{ij}\partial_i\left(J\frac{\delta L}{\delta v^j}\right) = \frac{1}{\sqrt{m}}\varepsilon^{ij}\partial_i\left(v_j + \sigma_j\right) \tag{4.20}$$

defines the potential vorticity q. The latter is conserved following fluid particles, i.e. $\mathrm{D}q/\mathrm{D}t = 0$, as readily follows upon noticing that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\oint_{\partial D}\mathrm{d}x^i\,J\frac{\delta L}{\delta v^i} &= \oint_{\partial D}\mathrm{d}x^i\left[\frac{\mathrm{D}}{\mathrm{D}t}\left(J\frac{\delta L}{\delta v^i}\right) + J\frac{\delta L}{\delta v^j}\partial_i v^j\right] \\ &= \oint_{\partial D}\mathrm{d}x^i\partial_i\frac{\delta L}{\delta\mathfrak{J}} \\ &\equiv 0. \end{aligned} \tag{4.21}$$

The physical counterpart of any of the equations in (4.18) follows from application of the inverse map $\mathbb{P}^{-1}$. In geographic coordinates, the physical counterpart of, for instance, set (4.18b), reads (e.g. R97)

$$\left.\begin{aligned} \partial_t u - (\xi + f)v + \gamma^{-1}\partial_x B = 0, \\ \partial_t v + (\xi + f)u + \partial_y B = 0, \end{aligned}\right\} \tag{4.22}$$

where $\xi := \gamma^{-1}\partial_x v - \partial_y u - \tau u$ and $B := p + \frac{1}{2}(u^2 + v^2)$ are the relative vorticity and Bernoulli head, respectively. System (4.22) can also be written in vector notation, i.e.

$$\partial_t\mathbf{u} + (\xi + f)\,\hat{\mathbf{z}}\times\mathbf{u} + \nabla B = 0, \tag{4.23}$$

with $\nabla a := (\gamma^{-1}\partial_x a, \partial_y a)$ the gradient of any scalar function $a(\mathbf{x})$ in geographic coordinates, and where $\xi = \nabla\cdot(\mathbf{u}\times\hat{\mathbf{z}})$ and $B = p + \frac{1}{2}\mathbf{u}^2$. Finally, the integrals of motion take the form

$$\mathcal{E} = \tfrac{1}{2}\int_D \mathrm{d}^2\mathbf{x}\,\gamma\left(h\mathbf{u}^2 + p^2/g\right), \quad \mathcal{C} = \int_D \mathrm{d}^2\mathbf{x}\,\gamma hC(q), \tag{4.24}$$

where $q = (\xi + f)/h$ satisfies

$$\mathrm{D}q/\mathrm{D}t = (\partial_t + u\gamma^{-1}\partial_x + v\partial_y)q = (\partial_t + \mathbf{u}\cdot\nabla)q = 0. \tag{4.25}$$

If D is a zonal channel and the topography has the same symmetry, i.e. $\partial_x H \equiv 0$, then zonal momentum,

$$\mathcal{M} := \int_D \mathrm{d}^2\mathbf{x}\, \gamma h \left[\gamma u - \Omega R \left(\cos\vartheta_0 - \gamma\cos\vartheta\right)\right], \tag{4.26}$$

is also an integral of motion.

4.5. HP FOR L1 DYNAMICS ON THE SPHERE

The starting point of S83's method to derive approximate models by making approximations in the HP for SW consists is expanding the velocity field as

$$\begin{array}{ccccc} \mathbf{v} & = & \mathbf{v}_\mathrm{G} & + & \mathbf{v}_\mathrm{A}, \\ O & : & \varepsilon & & \varepsilon^2 \end{array} \tag{4.27}$$

where $\varepsilon \to 0$ is an appropriate Rossby number. The lowest-order contribution to the velocity is assumed to satisfy the geostrophic balance and thus is a function of the height (mass) field. In invariant form this reads

$$\mathbf{v}_\mathrm{G}(\mathfrak{J}) = -(\mathbf{d}\sigma^\flat)^{-1} \cdot \mathbf{d}p \tag{4.28}$$

(at least there where $\mathbf{d}\sigma^\flat$ is invertible). The Lagrangian density for L1 dynamics on the sphere is obtained from (4.14) after replacing $\mathbf{v}$ by (4.27), with $\mathbf{v}_\mathrm{G}$ given by (4.28), and by dropping the $O(\varepsilon^4)$-term $\frac{1}{2}\|\mathbf{v}_\mathrm{A}\|^2$ in the first parenthesis. Thus

$$\boxed{l_1(\mathbf{v},\mathfrak{J};\mathbf{x}) := h_0\mathfrak{J}\left(\langle\langle \mathbf{v}, \mathbf{v}_\mathrm{G} + \sigma\rangle\rangle - \tfrac{1}{2}\|\mathbf{v}_\mathrm{G}\|^2\right) - \tfrac{1}{2} g\sqrt{m}\left(\frac{h_0\mathfrak{J}}{\sqrt{m}} - H\right)^2,} \tag{4.29}$$

together with the definitions (4.15), gives the L1 model's Lagrangian, i.e. $L_1 := \int_D \mathrm{d}^2\mathbf{x}\, l_1$. (A notation more consistent with my dimensional approach should in fact be L_3 for this Lagrangian.) According to

$$\frac{\delta L_1}{\delta \mathbf{v}} = h_0\mathfrak{J}\left(\mathbf{v}_\mathrm{G} + \sigma\right)^\flat, \tag{4.30}$$

$$\frac{\delta L_1}{\delta \mathfrak{J}} = h_0\left(\langle\langle \mathbf{v}, \mathbf{v}_\mathrm{G} + \sigma\rangle\rangle - \tfrac{1}{2}\|\mathbf{v}_\mathrm{G}\|^2 - p_\mathrm{AG}\right), \tag{4.31}$$

where $p_{\mathrm{AG}} := p - \mathrm{div}[gh(\mathbf{d}\sigma^\flat)^{-1}\cdot\mathbf{v}_{\mathrm{A}}^\flat]$. HP implies the following equivalent equations:

$$\boxed{\begin{aligned}&\text{a. } \partial_t(\mathbf{v}_{\mathrm{G}}+\sigma)^\flat + \boldsymbol{\pounds}_{\mathbf{v}}(\mathbf{v}_{\mathrm{G}}+\sigma)^\flat + \mathbf{d}\Big(p_{\mathrm{AG}} - \tfrac{1}{2}\|\mathbf{v}_G\|^2 - \langle\langle\mathbf{v},\mathbf{v}_{\mathrm{G}}+\sigma\rangle\rangle\Big) = 0,\\ &\text{b. } \partial_t\mathbf{v}_{\mathrm{G}}^\flat + \mathbf{d}(\mathbf{v}_{\mathrm{G}}+\sigma)^\flat\cdot\mathbf{v} + \mathbf{d}\Big(p_{\mathrm{AG}} + \tfrac{1}{2}\|\mathbf{v}_{\mathrm{G}}\|^2\Big) = 0,\\ &\text{c. } (\partial_t + \nabla_{\mathbf{v}})\mathbf{v}_{\mathrm{G}}^\flat + \mathbf{d}\mathbf{v}_{\mathrm{G}}^\flat\cdot\mathbf{v}_{\mathrm{A}} + \mathbf{d}\sigma^\flat\cdot\mathbf{v} + \mathbf{d}p_{\mathrm{AG}} = 0.\end{aligned}} \tag{4.32}$$

Because of the presence of the term $\langle\langle\mathbf{v},\mathbf{v}_{\mathrm{G}}\rangle\rangle$ in (4.29), in addition to the no-flow boundary condition (4.13), HP also implies the following condition:

$$\boxed{\langle\langle(\mathbf{d}\sigma^\flat)^{-1}\cdot\mathbf{v}_{\mathrm{A}}^\flat, n\rangle\rangle = 0 \quad @ \quad \partial D.} \tag{4.33}$$

Any set selected from (4.32) (or the corresponding contravariant counterpart through the metric) together with the volume conservation equation (4.9), all subject to boundary conditions (4.13) *and* (4.33), constitute the invariant form of the L1 model on a region D on the sphere. Since $\mathbf{v}_{\mathrm{G}}$ and h are not independent the L1 system has only one scalar prognostic equation; the other two scalar equations provide the constraints to determine $\mathbf{v}_{\mathrm{A}}$.

The L1 model conserves geostrophic versions of the SW energy and Casimirs, namely

$$\mathcal{E}_{\mathrm{G}} := \tfrac{1}{2}\int_D \mathrm{d}^2\mathbf{x}\,\sqrt{m}\left(h\,\|\mathbf{v}_{\mathrm{G}}\|^2 + p^2/g\right), \quad \mathcal{C}_{\mathrm{G}} := \int_D \mathrm{d}^2\mathbf{x}\,\sqrt{m}hC(q_{\mathrm{G}}) \tag{4.34}$$

for arbitrary $C(\cdot)$, where

$$q_{\mathrm{G}}h := \frac{1}{\sqrt{m}h_0}\varepsilon^{ij}\partial_i\left(J\frac{\delta L_1}{\delta v^j}\right) = \frac{1}{\sqrt{m}}\varepsilon^{ij}\partial_i\left(v_{\mathrm{G}j}+\sigma_j\right) \tag{4.35}$$

defines the geostrophic potential vorticity q_{G}, which is materially conserved as is advected by the *total* flow (i.e. $\mathrm{D}q_G/\mathrm{D}t = 0$).

In geographic coordinates, the physical counterpart of, for instance, set (4.32b) is given by

$$\left.\begin{aligned}\partial_t u_{\mathrm{G}} - (\xi_{\mathrm{G}}+f)v + \gamma^{-1}\partial_x B_{\mathrm{AG}} = 0,\\ \partial_t v_{\mathrm{G}} + (\xi_{\mathrm{G}}+f)u + \partial_y B_{\mathrm{AG}} = 0,\end{aligned}\right\} \tag{4.36}$$

where $\xi_{\mathrm{G}} := \gamma^{-1}\partial_x v_{\mathrm{G}} - \partial_y u_{\mathrm{G}} - \tau u_{\mathrm{G}},\ B_{\mathrm{AG}} := p_{\mathrm{AG}} + \frac{1}{2}(u_{\mathrm{G}}^2 + v_{\mathrm{G}}^2)$, and $p_{\mathrm{AG}} = p + \gamma^{-1}\partial_x(ghv_{\mathrm{A}}/f) - \partial_y(ghu_{\mathrm{A}}/f) - \tau ghu_{\mathrm{A}}/f$. In vector notation set (4.36) expresses as

$$\partial_t\mathbf{u}_{\mathrm{G}} + (\xi_{\mathrm{G}}+f)\,\hat{\mathbf{z}}\times\mathbf{u} + \nabla B_{\mathrm{AG}} = 0, \tag{4.37}$$

where $\xi_{\rm G} = \nabla\cdot(\mathbf{u}_{\rm G}\times\hat{\mathbf{z}})$, $B_{\rm AG} = p_{\rm AG}+\frac{1}{2}\mathbf{u}_{\rm G}^2$, and $p_{\rm AG} = p+\nabla\cdot(gh\mathbf{u}_{\rm A}\times\hat{\mathbf{z}}/f)$. Boundary condition (4.33), in turn, takes the form

$$\mathbf{u}_{\rm A}\cdot\hat{\mathbf{z}}\times\hat{\mathbf{n}} = 0 \quad @ \quad \partial D \tag{4.38}$$

(cf. Ren and Shepherd, 1997 for a physical interpretation of this condition). The set of diagnostic equations which determines $\mathbf{u}_{\rm A}$ is given by

$$\left.\begin{aligned} \mathsf{A}(hv_{\rm A}) + \mathsf{B}(hu_A) &= F_1,\\ \mathsf{A}((g/f)hv_{\rm A}) + (g/f)\mathsf{B}(hu_A) &= F_2,\end{aligned}\right\} \tag{4.39}$$

where the differential operators

$$\mathsf{A}(\cdot) := \nabla^2(\cdot) - R^{-2} - \tau^2 - (f/g)q_{\rm G}, \quad \mathsf{B}(\cdot) := (f'/f)\gamma^{-1}\partial_x(\cdot), \tag{4.40}$$

and the functions

$$F_1(h) := -\partial_y\nabla\cdot(h\mathbf{u}_{\rm G}) + (f/g)\left(hq_{\rm G}v_{\rm G} - \gamma^{-1}\partial_x B_{\rm G}\right), \tag{4.41}$$

$$F_2(h) := -(g/f)\gamma^{-1}\partial_x\nabla\cdot(h\mathbf{u}_{\rm G}) + hq_{\rm G}u_{\rm G} + \partial_y B_{\rm G}; \tag{4.42}$$

here, $\nabla^2 a := \nabla\cdot\nabla a = \gamma^{-2}\partial_{xx}a + \partial_{yy}a - \tau\partial_y a$ is the Laplacian of any scalar function $a(\mathbf{x})$ in geographic coordinates*. For completeness, from (4.39) it follows

$$hu_A = \left(\mathsf{A}(g/f) - (g/f)\mathsf{B}\mathsf{A}^{-1}\mathsf{B}\right)^{-1}\left(F_2 - (g/f)\mathsf{B}\mathsf{A}^{-1}F_1\right), \tag{4.43a}$$

$$hv_A = \left(\mathsf{A} - \mathsf{B}(f/g)\mathsf{A}^{-1}(g/f)\mathsf{B}\right)^{-1}\left(F_1 - \mathsf{B}(f/g)\mathsf{A}^{-1}F_2\right), \tag{4.43b}$$

which upon substitution in the volume conservation equation (4.9) results in a single evolution equation for the height field. The (Cartesian) f-plane version of the latter was derived by Vanneste and Bokhove (2002) using a Dirac-bracket approach. Finally, the integrals of motion of the L1 system read

$$\mathcal{E}_{\rm G} := \tfrac{1}{2}\int_D \mathrm{d}^2\mathbf{x}\,\gamma\left(h\mathbf{u}_{\rm G}^2 + p^2/g\right), \quad \mathcal{C}_{\rm G} := \int_D \mathrm{d}^2\mathbf{x}\,\gamma h C(q_{\rm G}), \tag{4.44}$$

where $q_{\rm G} = (\xi_{\rm G} + f)/h$; as before if D and H are zonally symmetric then

$$\mathcal{M}_{\rm G} := \int_D \mathrm{d}^2\mathbf{x}\,\gamma h\left[\gamma u_{\rm G} - \Omega R\left(\cos\vartheta_0 - \gamma\cos\vartheta\right)\right] \tag{4.45}$$

*Because $\gamma^2 > 0$ (excluding, of course, the poles) the elliptic problem (4.43) has a unique solution on D (bounded or periodic in one or both directions) provided that $q_{\rm G}f \geq -g(R^{-2}+\tau^2)$ (cf. Courant and Hilbert, 1962), which holds for all time because $q_{\rm G} \sim f/h$ as $\varepsilon \to 0$.

is also conserved.

Other decompositions, appart than (4.27), as well as other balance relationships, different than (4.28), are possible (Allen and Holm, 1996; Allen et al., 2002). This freedom is what allows for the existence of approximate models which can be potentially more accurate than the L1 model.

5. Concluding Remarks

The scaling

$$\{\mathbf{u}, \partial_t, y/R, h - H(y), H'\} = O(\varepsilon) \tag{5.46}$$

implies, at $O(\varepsilon^2)$, the classical QG equation (cf. Pedlosky, 1987)

$$(\partial_t + \partial_x\psi\partial_y - \partial_y\psi\partial_x)\, q_{\mathrm{QG}} = 0, \tag{5.47a}$$

where

$$q_{\mathrm{QG}} := \left[\partial_{xx} + \partial_{yy} - f_0^2/(gH_0)\right]\psi + (\beta + \beta_{\mathrm{T}})\, y. \tag{5.47b}$$

Here, $H(y) = H_0(1 - \beta_{\mathrm{T}} y/f_0)$, with $H_0 = \mathrm{const}$., and $\psi := g[h - H(y)]/f_0$ is the geostrophic streamfunction, i.e. $\mathbf{u} = (-\partial_y\psi, \partial_x\psi) + O(\varepsilon^2)$. (More complicated topographies can of course be considered.) Notice the absence of geometric coefficients in (5.47). Those terms, which *do* appear in the corresponding (diagnostic) momentum and volume conservation equations, have (fortuitously) cancelled out in the process of constructing the (prognostic) potential vorticity equation (5.47) [Pedlosky 1987; R97]. Consequently—and remarkably—QG flows develop *as if* the geometry were Cartesian, "feeling" the latitudinal variation of the Coriolis parameter as the *only* effect of the Earth's sphericity.

The L1 model shares a series of differences and similarities with the above QG model. Although both models are derivable from HP, the QG model's action is not seen to derive from approximations performed in SW's action. As QG motions, those governed by the L1 model are not allowed at the equator, i.e. where f vanishes. In addition to Rossby waves, the linear waves of the L1 model include (a form of) Kelvin waves, which are not supported by the QG model. Unlike QG motions, L1 motions are restricted neither to $O(\varepsilon)$ meridional excursions nor to $O(\varepsilon)$ displacements of the free surface from the position of equilibrium at rest, nor to the presence of $O(\varepsilon)$ topographic variations. In a reduced-gravity setting, the equations for both SW and L1 models have the same structure as those presented here, except that in that case g must be identified with the buoyancy jump at the interface between the active and the quiescent (infinitely deep) bottom layer, and $H(\mathbf{x})$ must be understood as the nonuniform thickness of the active layer at rest, including the possibility of a nonspherical rigid surface.

Consequently, in contrast to the QG model, the L1 model is able to describe the dynamics of frontal structures.

The integrals of motion of the L1 system, in geographic coordinates, expand in inverse powers of the radius of the spherical Earth R as

$$\mathcal{E}_{\mathrm{G}}=\left\langle\left(\frac{1}{2}-\frac{\beta y}{f_0}\right)\frac{(\partial_x p)^2+(\partial_y p)^2}{f_0^2}+\frac{\tau_0 y}{f_0^2}(\partial_x p)^2+\frac{p^2}{2gh}\right\rangle \tag{5.48}$$

$$\mathcal{M}_{\mathrm{G}}=\left\langle\left(\frac{2\tau_0 f_0+\beta}{f_0}y-1\right)\frac{\partial_y p}{f_0}-(1-\tau_0 y)\,f_0 y-\frac{1}{2}\beta\left(1-R^2\tau_0^2\right)y^2\right\rangle \tag{5.49}$$

$$\mathcal{C}_{\mathrm{G}}=\langle(1-\tau_0 y)C(q_0)+q_1 C'(q_0)\rangle \tag{5.50}$$

$+\,O(R^{-2})$. Here, $\langle\cdot\rangle := \int_D \mathrm{d}^2\mathbf{x}\,h\,(\cdot)$, and

$$q_0 h := \frac{\partial_{xx}p+\partial_{yy}p}{f_0}+f_0, \tag{5.51}$$

$$q_1 h := \frac{(2\tau_0 f_0-\beta)\partial_{xx}p-\beta\partial_{yy}p}{f_0^2}y+\frac{\tau_0 f_0-\beta}{f_0^2}\partial_y p+\beta y. \tag{5.52}$$

Clearly, a consistent (not necessarily the optimal, though) geometric approximation for L1 dynamics, which is first-order accurate in R, is given by the non-Cartesian Ripa "plane" and *not* by the standard β plane (recall that it has $\tau_0 = 0$). The latter is the one included in the original derivation of the L1 model. An important contribution of the present work to the above list of differences and similarities between the L1 and QG models is thus the sensitivity of the former to the differences between geographic and geodesic coordinates. This result confirms Ripa's (2000b) in the sense that Earth's curvature effects increase in importance as the motions deviate from strictly geostrophic (divergence-free) motions. The thorough evaluation of these effects, apart from checking that the equations have the right conservation laws, is a subject for futher research. The latter should involve direct numerical simulations in which predictions of the L1 model on the β-plane and the sphere (or the Ripa "plane") are compared.

Finally, Ripa (1983) showed that steady SW flows on the sphere posses a formal stability theorem. The latter involving an Arnold-like first theorem for the stability of QG flows, and a condition for the flow to be "subsonic" in the sense that the (geostrophic) basic flow must be everywhere slower than the slowest gravity-wave of the system. Ren and Shepherd (1997) showed, in turn, that steady L1 flows on the β plane posses a Ripa-like formal stability theorem, as well as a nonlinear (or Lyapunov) stability theorem in which the "subsonic" condition of Ripa's theorem is replaced by a condition that the flow be cyclonic along the lateral boundaries. The latter was shown by Ren and Shepherd (1997) to have an interpretation involving coastal Kelvin waves, which are not included in the QG model. Whether or not L1 flows on the sphere (or the Ripa “plane”) enjoy similar stability properties is an issue that needs more investigation.

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A. Useful Relations

The following relationships can be shown to hold:

$$J\mathfrak{J} = 1 \Longleftrightarrow J^i_{\mathfrak{i}}\mathfrak{J}^{\mathfrak{i}}_j = \delta^i_j, \tag{A.1a}$$

$$(\operatorname{adj} J)^{\mathfrak{i}}_i = J\mathfrak{J}^{\mathfrak{i}}_i \Longleftrightarrow J = J^i_{\mathfrak{i}}(\operatorname{adj} J)^{\mathfrak{i}}_i \Longleftrightarrow \partial J/\partial J^i_{\mathfrak{i}} = J\mathfrak{J}^{\mathfrak{i}}_i, \tag{A.1b}$$

$$(\operatorname{adj} \mathfrak{J})^i_{\mathfrak{i}} = \mathfrak{J}J^i_{\mathfrak{i}} \Longleftrightarrow \mathfrak{J} = \mathfrak{J}^{\mathfrak{i}}_i(\operatorname{adj} \mathfrak{J})^i_{\mathfrak{i}} \Longleftrightarrow \partial \mathfrak{J}/\partial \mathfrak{J}^{\mathfrak{i}}_i = \mathfrak{J}J^i_{\mathfrak{i}}, \tag{A.1c}$$

$$\partial_{\mathfrak{i}}\left(J\mathfrak{J}^{\mathfrak{i}}_i\right) = 0 = \partial_i\left(\mathfrak{J}J^i_{\mathfrak{i}}\right), \tag{A.1d}$$

$$\delta J = J\partial_i\delta x^i,\ \delta\mathfrak{J} = \mathfrak{J}\partial_{\mathfrak{i}}\delta\mathfrak{x}^{\mathfrak{i}}, \tag{A.1e}$$

$$\partial_{\mathfrak{i}} = J^i_{\mathfrak{i}}\partial_i,\ \partial_i = \mathfrak{J}^{\mathfrak{i}}_i\partial_{\mathfrak{i}}. \tag{A.1f}$$

In deriving (A.1e) the following properties of the determinants were very helpful

$$\frac{\partial(a,b)}{\partial(\mathfrak{x}^1,\mathfrak{x}^2)} = \frac{\partial(a,b)}{\partial(x^1,x^2)}\frac{\partial(x^1,x^2)}{\partial(\mathfrak{x}^1,\mathfrak{x}^2)}, \quad \frac{\partial(a,x^2)}{\partial(x^1,x^2)} = \frac{\partial a}{\partial x^1} \tag{A.2}$$

for all scalar functions $a, b(\mathbf{x})$.

In addition, it can be shown that:

$$\dot{\mathfrak{x}}^{\mathfrak{i}} = \partial_t\mathfrak{x}^{\mathfrak{i}} + v^i\mathfrak{J}^{\mathfrak{i}}_i = 0 \Longrightarrow v^i = -\mathfrak{J}^i_{\mathfrak{i}}\partial_t\mathfrak{x}^{\mathfrak{i}}, \tag{A.3a}$$

$$\partial_{\mathfrak{i}}v^i = J^j_{\mathfrak{i}}\partial_j v^i \Longrightarrow \dot{J}^i_{\mathfrak{i}} = J^j_{\mathfrak{i}}\partial_j v^i, \tag{A.3b}$$

$$\delta v^i = -J^i_{\mathfrak{i}}(\partial_t + v^j\partial_j)\delta\mathfrak{x}^{\mathfrak{i}}, \quad \delta\mathfrak{J} = \mathfrak{J}J^i_{\mathfrak{i}}\partial_i\delta\mathfrak{x}^{\mathfrak{i}}, \tag{A.3c}$$

$$\delta J = \mathfrak{J}\partial_i\delta x^i \Longrightarrow \partial_t\mathfrak{J} + \partial_i\left(\mathfrak{J}v^i\right) = 0. \tag{A.3d}$$

B. Alternative HPs

B.1. EULERIAN COORDINATES

The standard approach for fields $\mathfrak{x}(\mathbf{x}, t)$ consists in considering an action functional of the form

$$\mathcal{S}[\mathfrak{x}] := \int_{t_0}^{t_1} \mathrm{d}t\, L[\mathfrak{x}, \partial_t \mathfrak{x}] = \int_{t_0}^{t_1} \mathrm{d}t \int_D \mathrm{d}^2\mathbf{x}\, l(\mathfrak{x}, \partial_t \mathfrak{x}, \mathfrak{J}^{\mathfrak{i}}_i; \mathbf{x}, t), \tag{B.1}$$

which, after invoking HP results in the familiar Euler–Lagrange equations

$$\partial_t \frac{\delta L}{\delta \mathfrak{x}^{\mathfrak{i}}_{,t}} - \frac{\delta L}{\delta \mathfrak{x}^{\mathfrak{i}}} = \partial_t \frac{\partial l}{\partial \mathfrak{x}^{\mathfrak{i}}_{,t}} + \partial_i \frac{\partial l}{\partial \mathfrak{J}^{\mathfrak{i}}_i} - \partial_{\mathfrak{i}} l = 0 \tag{B.2}$$

(plus boundary conditions). According to $\partial_i(\partial l/\partial \mathfrak{J}^{\mathfrak{i}}_i) = \partial_i(\mathfrak{J} J^i_{\mathfrak{i}} \partial l/\partial \mathfrak{J}) = \mathfrak{J} J^i_{\mathfrak{i}} \partial_i(\partial l/\partial \mathfrak{J})$ and (A.3a), equations (B.2) transform into

$$\partial_t \left(J^i_{\mathfrak{i}} \frac{\partial l}{\partial v^i} \right) - \mathfrak{J} J^i_{\mathfrak{i}} \partial_i \frac{\partial l}{\partial \mathfrak{J}} + \partial_{\mathfrak{i}} l = 0. \tag{B.3}$$

The latter can be shown to be equivalent to (4.12) only in the particular case $\partial_{\mathfrak{i}} l \equiv 0$.

B.2. LAGRANGIAN COORDINATES

In the variational approach for fields $\mathbf{x}(\mathfrak{x}, t)$ the action is of the form

$$\mathcal{S}[\mathbf{x}] := \int_{t_0}^{t_1} \mathrm{d}t\, L[\mathbf{x}, \partial_t \mathbf{x}] = \int_{t_0}^{t_1} \mathrm{d}t \int_{\mathfrak{D}} \mathrm{d}^2\mathfrak{x}\; l(\mathbf{x}, \partial_t \mathbf{x}, J^i_{\mathfrak{i}}; \mathfrak{x}, t). \tag{B.4}$$

Upon variations of particle paths at fixed Lagrangian labels and time, the following Euler–Lagrange equations result from HP:

$$\partial_t \frac{\delta L}{\delta x^i_{,t}} - \frac{\delta L}{\delta x^i} = \partial_t \frac{\partial l}{\partial x^i_{,t}} + \partial_{\mathfrak{i}} \frac{\partial l}{\partial J^i_{\mathfrak{i}}} - \partial_i l = 0 \tag{B.5}$$

(plus appropriate boundary conditions). This is but the infinite-dimensional analogue of the particle's HP. For instance, S83's derivation of the SW and L1 systems in the Cartesian coordinates of the β plane and van der Toorn's (1997) derivation of the SW equations on the sphere are based on this HP. One disadvantage of this variational approach, however, is that the resulting equations are in Lagrangian coordinates, which requires application of the inverse map φ^{-1} (4.2) to transform back to Eulerian coordinates.

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