

A Didactic Approach to Linear Waves in the Ocean

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The general equations of motion for ocean dynamics are presented and the waves supported by the (inviscid, unforced) linearized system with respect to a state of rest are derived. The linearized dynamics sustains one zero frequency mode (called buoyancy mode) in which salinity and temperature rearrange in such a way that seawater density does not change. Five nonzero frequency modes (two acoustic modes, two inertia-gravity or Poincaré modes, and one planetary or Rossby mode) are also sustained by the linearized dynamics, which satisfy an asymptotic general dispersion relation. The most usual approximations made in physical oceanography (namely incompressibility, Boussinesq, hydrostatic, and quasigeostrophic) are also considered, and their implications in the reduction of degrees of freedom (number of independent dynamical fields or prognostic equations) of, and compatible waves with, the linearized governing equations are particularly discussed and emphasized.

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I. INTRODUCTION

The goal of this educational work is to show how the various types of linear waves in the ocean (acoustic, inertia-gravity or Poincaré, and planetary or Rossby waves) can be obtained from a general dispersion relation in an approximate (asymptotic) sense. Knowledge of the theory of partial differential equations, and basic classical and fluid mechanics are only needed for the reader to understand the material presented here, which could be taught as a special topic in a course of fluid mechanics for physicists.

The exposition starts by presenting the general equations of motion for ocean dynamics in Sec. II. This presentation is not intended to be rigorous, but rather conceptual. Accordingly, the equations of motion are simplified as much as possible for didactic purposes. The general dispersion relation for the waves supported by the (inviscid, unforced) linearized dynamics with respect to a quiescent state is then derived in Sec. III. This is done by performing a separation of variables between the vertical direction, on one side, and the horizontal position and time, on the other side. Sec. IV discusses the implications of the most common approximations made in oceanography (namely incompressibility, Boussinesq, hydrostatic, and quasigeostrophic) in the reduction of degrees of freedom (number of independent dynamical fields or prognostic equations) of, and compatible waves with, the linearized governing equations. Particular emphasis is made on this important issue, which is vaguely covered in standard textbooks (e.g. Refs. 2,3,5). Some problems have been interspersed within the text to help the reader to assimilate the material presented. The solutions to some of these problems are outlined in App. A.

II. GENERAL EQUATIONS OF MOTION

Let $\mathbf{x} := (x, y)$ be the horizontal position, i.e. tangential to the Earth's surface, with x and y its eastward and northward components, respectively; let z be the upward coordinate; and let t be time. Unless otherwise stated all variables are functions of (\mathbf{x}, z, t) in this paper.

The *thermodynamic state* of the ocean is determined by three variables, typically S (salinity), T (temperature), and p (pressure, referred to one atmosphere). Seawater density, ρ , is a function of these three variables, i.e. $\rho = \rho(S, T, p)$, known as the state equation of seawater. In particular,

$$\rho^{-1}D\rho = \alpha_S DS - \alpha_T DT + \alpha_p Dp. \quad (1)$$

Here, $D := \partial_t + \mathbf{u} \cdot \nabla + w\partial_z$ is the substantial or material derivative, where \mathbf{u} and w are the horizontal and vertical components of the velocity field, respectively, and ∇ denotes the horizontal gradient; $\alpha_S := \rho^{-1}(\partial_S \rho)_{T,p}$ and $\alpha_T := \rho^{-1}(\partial_T \rho)_{S,p}$ are the haline contraction and thermal expansion coefficients, respectively; and $\alpha_p := \rho^{-1}(\partial_p \rho)_{T,S} = \alpha_T \Gamma + \rho^{-1}c_s^{-2}$, where Γ is the adiabatic gradient and c_s is the speed of sound, which characterize the compressibility of seawater.

The *physical state* of the ocean is determined at every instant by the above three variables (S, T, p) and the three components of the velocity field (\mathbf{u}, w) , i.e. *six* independent scalar variables. The evolution of these variables are controlled by

$$DS = F_S, \quad (2a)$$

$$DT = \Gamma Dp + F_T, \quad (2b)$$

$$Dp = -\alpha_p^{-1}(\nabla \cdot \mathbf{u} + \partial_z w + \alpha_T F_T - \alpha_S F_S), \quad (2c)$$

$$D\mathbf{u} = -\rho^{-1}\nabla p + \mathbf{F}_u, \quad (2d)$$

$$Dw = -\rho^{-1}\partial_z p - g + F_w. \quad (2e)$$

In Newton's horizontal equation (2d), the term \mathbf{F}_u represents the acceleration due to the horizontal

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components of the Coriolis and frictional forces. In Newton's vertical equation (2d), the term F_w represents the acceleration due to the vertical component of the Coriolis and frictional forces, and g is the (constant) acceleration due to gravity. The term F_S in the salinity equation (2a) represents diffusive processes. The term F_T in the thermal energy equation (2b), which follows from the first principle of thermodynamics, represents the exchange of heat by conduction and radiation, as well as heating by change of phase, chemical reactions or viscous dissipation. The pressure or continuity equation (2c) follows from (1).

Problem 1 Investigate why (2d,e) do not include the centrifugal force which would also be needed to describe the dynamics in a noninertial reference frame such as one attached to the Earth.

Since adiabatic compression does not have important dynamical effects, in physical oceanography it is commonly neglected. This is accomplished upon introducing the potential temperature θ , which satisfies $\alpha_\theta D\theta = \alpha_T(DT - \Gamma Dp)$, so that $\rho = \rho(S, \theta, p)$ and (1) is consistently replaced by

$$\rho^{-1} D\rho = \alpha_S DS - \alpha_\theta D\theta + \rho^{-1} c_s^{-2} Dp, \quad (3)$$

where here it must be understood that $\alpha_S = (\partial_S \rho)_{\theta, p}$ and $\alpha_T = (\partial_\theta \rho)_{S, p}$. Equations (2b,c) then are replaced, respectively, by

$$D\theta = F_\theta, \quad (4a)$$

$$Dp = -\rho c_s^2 (\nabla \cdot \mathbf{u} + \partial_z w + \alpha_\theta F_\theta - \alpha_S F_S). \quad (4b)$$

As our interest is in the waves sustained by the linearized dynamics, we do not need to consider either diffusive processes or allow the motion to depart from isentropic. Hence, we will set $F_S \equiv 0 \equiv F_\theta$ so that equations (2a) and (4) can be substituted, respectively, by

$$D\zeta = w, \quad (5a)$$

$$Dp = -\rho c_s^2 (\nabla \cdot \mathbf{u} + \partial_z w), \quad (5b)$$

where ζ is the vertical displacement of an isopycnal which is defined such that $\rho = \rho_r(z - \zeta)$.

Problem 2 Show that equations (2a) and (4a) certainly lead to (5a) when $F_S \equiv 0 \equiv F_\theta$.

We will also neglect frictional effects, so that equations (2d,e) are seen to be nothing but Euler equations of (ideal) fluid mechanics with the addition of the Coriolis force. The latter will be further considered as due solely to the vertical component of the Earth rotation. Thus, the following simplified form of equations (2d,e) will be considered:

$$D\mathbf{u} = -\rho^{-1} \nabla p - f \hat{\mathbf{z}} \times \mathbf{u}, \quad (6a)$$

$$Dw = -\rho^{-1} \partial_z p - g. \quad (6b)$$

Here, $\hat{\mathbf{z}}$ is the upward unit vector and $f := 2\Omega \sin \vartheta$, where Ω is the (assumed constant) spinning rate of the Earth around its axis and ϑ is the geographical latitude, is the Coriolis parameter. For simplicity, we will avoid working in full spherical geometry. Instead, we will consider $f = f_0 + \beta y$, where $f_0 := 2\Omega \sin \vartheta_0$ and $\beta := 2\Omega R^{-1} \cos \vartheta_0$ with ϑ_0 a fixed latitude and R the mean radius of the planet, and $\nabla = (\partial_x, \partial_y)$, which is known as the β -plane approximation. It should remain clear, however, that a consistent β -plane approximation must include some geometric (non-Cartesian) terms.⁷ Neither these terms nor those of the Coriolis force due to the horizontal component of the Earth's rotation contribute to add waves to the linearized equations of motion. Their neglect is thus well justified for the purposes of this paper.

III. WAVES OF THE LINEARIZED DYNAMICS

Consider a state of rest ($\mathbf{u} = \mathbf{0}$, $w = 0$) characterized by $dp_r/dz = -\rho_r g$, where $p_r(z)$ and $\rho_r(z)$ are reference profiles of pressure and density, respectively. Equations (4) and (6), linearized with respect to that state, can be written as

$$\partial_t \zeta' = w', \quad (7a)$$

$$\partial_t p' = -\rho_r c_s^2 (\nabla \cdot \mathbf{u}' + \partial_z^- w'), \quad (7b)$$

$$\partial_t \mathbf{u}' = -\rho_r^{-1} \nabla p' - f \hat{\mathbf{z}} \times \mathbf{u}', \quad (7c)$$

$$\partial_t w' = -\rho_r^{-1} \partial_z^+ p' - N^2 \zeta'. \quad (7d)$$

Here, primed quantities denote perturbations with respect to the state of rest; $\partial_z^\pm := \partial_z \pm g c_s^{-2}$; $N^2(z) := -g(\rho_r^{-1} d\rho_r/dz + g c_s^{-2})$ is the square of the reference Brunt-Väisälä frequency; and c_s is assumed constant. In addition to the above equations, it is clear that

$$\partial_t S' = -w dS_r/dz, \quad (8a)$$

$$\partial_t \theta' = -w d\theta_r/dz, \quad (8b)$$

where $S_r(z)$ and $\theta_r(z)$ are reference salinity and potential density profiles, respectively.

Problem 3 Work out the linearization of the equations of motion.

A. Zero Frequency Mode

The linearized dynamics supports a solution with vanishing frequency ($\partial_t \equiv 0$) such that

$$\zeta' \equiv 0, \quad p' \equiv 0, \quad \mathbf{u}' \equiv \mathbf{0}, \quad w' \equiv 0, \quad (9)$$

as it follows from (7), but with

$$S' \neq 0, \quad \theta' \neq 0, \quad (10)$$

as can be inferred from (8). Namely for this solution the salinity and temperature fields vary without changing the density of the fluid. More precisely, one has, on one hand, $\rho' = \rho_r(g^{-1}N^2 + gc_s^{-2})\zeta'$, and, on the other, $\rho' = \rho_r(\alpha_S S' - \alpha_\theta \theta' + \alpha_p p')$, where the α 's are evaluated at the reference state. By virtue of (9) then it follows that

$$\alpha_S S' - \alpha_\theta \theta' = 0. \quad (11)$$

This so-called *buoyancy mode* describes small scale processes in the ocean such as double diffusion.

B. Nonzero Frequency Modes

Upon eliminating w between (7a) and (7d), and proposing a separation of variables between z , on one side, and (\mathbf{x}, t) , on the other side, for the horizontal velocity and pressure fields in the form

$$\mathbf{u}' = \mathbf{u}^c(\mathbf{x}, t) \partial_z^- F(z), \quad p' = \rho_r(z) p^c(\mathbf{x}, t) \partial_z^- F(z), \quad (12)$$

it follows that

$$c_s^{-2} \partial_z^- F \partial_t p^c + (\partial_z^- F \nabla \cdot \mathbf{u}^c + \partial_{zt}^- \zeta') = 0, \quad (13a)$$

$$\partial_t \mathbf{u}^c + f \hat{\mathbf{z}} \times \mathbf{u}^c + \nabla p^c = \mathbf{0}, \quad (13b)$$

$$\partial_{tt} \zeta' + \rho_r^{-1} \partial_z^+ (\rho_r \partial_z^- F) p^c + N^2 \zeta' = 0. \quad (13c)$$

Now, assuming a common temporal dependence of the form $e^{-i\omega t}$, from (13c) one obtains

$$\zeta' = -\frac{p^c \partial_z^+ (\rho_r \partial_z^- F)}{\rho_r (N^2 - \omega^2)}. \quad (14)$$

Then, upon substituting (14) in (13a) it follows that

$$c_s^{-2} - \frac{1}{\partial_z^- F} \partial_z^- \left[\frac{\partial_z^+ (\rho_r \partial_z^- F)}{\rho_r (N^2 - \omega^2)} \right] = -\frac{\nabla \cdot \mathbf{u}^c}{\partial_t p^c} = c^{-2}, \quad (15)$$

where c is a constant known as the *separation constant*. Clearly, c_s must be chosen as a constant in order for the separation of variables to be possible. From (15) it follows, on one hand, that

$$\partial_z^+ (\rho_r \partial_z^- F) + \rho_r (N^2 - \omega^2) (c^{-2} - c_s^{-2}) F = 0, \quad (16)$$

and taking into account (13b), it follows, on the other hand, that

$$\partial_t p^c + c^2 \nabla \cdot \mathbf{u}^c = 0, \quad (17a)$$

$$\partial_t \mathbf{u}^c + f \hat{\mathbf{z}} \times \mathbf{u}^c + \nabla p^c = \mathbf{0}. \quad (17b)$$

Equation (16) governs the *vertical structure* of the perturbations, whereas system (17) controls the evolution of their *horizontal structure*.

Problem 4 Show that the alternate separation of variables which uses $F(z)$ instead of $\partial_z^- F(z)$ leads to a vertical structure equation with a singularity where $\omega^2 \equiv N^2$.

Equation (16) can be presented in different forms according to the approximation performed. Under the *incompressibility approximation*, which consists of making the replacement $\partial_t p' + gw' \mapsto 0$ in the continuity equation (7b), equation (16) takes the form

$$\partial_z^+ (\rho_r \partial_z F) + \rho_r c^{-2} (N^2 - \omega^2) F = 0. \quad (18)$$

This approximation corresponds formally to taking the limit $c_s^{-2} \rightarrow 0$. The *hydrostatic approximation*, in turn, consists of making the replacement $\partial_t w' \mapsto 0$ in Newton's vertical equation (7d). This way, without the need of assuming any particular temporal dependence, it follows that $\zeta' = -p^c \partial_z^+ (\rho_r \partial_z^- F) / (\rho_r N^2)$. Consequently, equation (16) reduces to

$$\partial_z^+ (\rho_r \partial_z^- F) + \rho_r (c^{-2} - c_s^{-2}) N^2 F = 0. \quad (19)$$

This approximation is valid for $\omega^2 \ll N^2$, i.e. periods exceeding the local buoyancy period which typically is of about 1 h. (Of course, this approximation implies that of incompressibility as it filters out the acoustic modes whose frequencies are much higher than the Brunt-Väisälä frequency.) Another common approximation is the *Boussinesq approximation*, which consists of making the replacements $\rho_r \mapsto \bar{\rho} = \text{const.}$ and $\partial_z^\pm \mapsto \partial_z$ in (7). Under this approximation, equation takes the simpler form

$$d^2 F/dz^2 + (c^{-2} - c_s^{-2}) (N^2 - \omega^2) F = 0. \quad (20)$$

Problem 5 Show that the Boussinesq approximation is very good for the ocean but not so for the atmosphere. Hint: This approximation requires $c_s^2 \gg gH$, where H is a typical vertical length scale.

1. Horizontal Structure

To describe these waves is convenient to introduce a potential $\varphi(\mathbf{x}, t)$ such that^{6,9}

$$p^c = -c^2 (\partial_{ty} + f \partial_x) \varphi, \quad (21a)$$

$$u^c = (c^2 \partial_{xy} + f \partial_t) \varphi, \quad (21b)$$

$$v^c = (\partial_{tt} - c^2 \partial_{xx}) \varphi, \quad (21c)$$

which allows one to reduce system (17) to a single equation in one variable:

$$\mathcal{L} \varphi := \{ \partial_t [\partial_{tt} + f^2(y) - c^2 \nabla^2] - \beta c^2 \partial_x \} \varphi = 0. \quad (22)$$

The linear differential operator \mathcal{L} contains a variable coefficient and, hence, a solution to (22) must be of the form

$$\varphi = \Phi(y) e^{i(kx - \omega t)} \quad (23)$$

with $\Phi(y)$ satisfying

$$d^2 \Phi/dy^2 + l^2(y) \Phi = 0 \quad (24)$$

where

$$l^2(y) := -k^2 - \beta \frac{k}{\omega} + \frac{\omega^2 - f^2(y)}{c^2}. \quad (25)$$

Now, if $l^2(y)$ is positive and sufficiently large, then $\Phi(y)$ oscillates like

$$\Phi(y) \sim e^{\pm i \int^y dy l(y)}. \quad (26)$$

This is known as the WKB approximation (cf. e.g. Ref. 4), where $l(y)$ defines a local meridional wavenumber in the approximate (asymptotic) dispersion relation

$$\omega^2 - (f^2 + c^2 \mathbf{k}^2) - \beta \frac{k}{\omega} = 0, \quad (27)$$

where $\mathbf{k} := (k, l)$ is the horizontal wavenumber.

System (17) also supports a type of nondispersive waves called Kelvin waves. These waves have $v^c \equiv 0$ and thus are seen to satisfy

$$\partial_t p^c + c^2 \partial_x u^c = 0, \quad (28a)$$

$$\partial_t u^c + \partial_x p^c = 0, \quad (28b)$$

$$f u^c + \partial_y p^c = 0. \quad (28c)$$

Clearly, these waves propagate as nondispersive waves in the zonal (east–west) direction—as if it were $f \equiv 0$ —and are in geostrophic balance between the Coriolis and pressure gradient forces in the meridional (south–north) direction. From (28a, b) it follows that

$$p^c = A(y)K(x - ct) \equiv cu^c, \quad (29)$$

where $K(\cdot)$ is an arbitrary function. By virtue of (28c) then it follows $dA/dy + fA/c = 0$, whose solution is

$$A(y) \propto e^{-\int^y dy f(y)/c} = e^{-(f_0 y + \frac{1}{2} \beta y^2)/c},$$

which requires, except there where $f_0 \equiv 0$ (i.e. the equator), the presence of a zonal coast to be physically meaningful.

Problem 6 Consider the Kelvin waves in the so-called f plane, i.e. with $\beta \equiv 0$.

2. Vertical Structure

Under the Boussinesq approximation the five fields of system (7) remain independent, thereby removing no wave solutions. We can thus safely consider the vertical structure equation (20), which we rewrite in the form

$$d^2 F/dF^2 + m^2(z)F = 0 \quad (30)$$

where

$$m^2(z) := [N^2(z) - \omega^2] (c^{-2} - c_s^{-2}). \quad (31)$$

Equation (30) can be understood in two different senses. Within the realm of the WKB approximation, (31) defines a local vertical wavenumber, and a solution to (30) oscillates like

$$F(z) \sim e^{\pm i \int^z dz m(z)}. \quad (32)$$

The other sense is that of *vertical normal modes*, in which (30) is solved in the whole water column with boundary conditions

$$F(-H) = 0, \quad (33a)$$

$$gF(0) = c^2 dF(0)/dz. \quad (33b)$$

Condition (33a) comes from imposing $w' = 0$ at $z = -H$ where H , which must be a constant, is the depth of the fluid in the reference state. Condition (33b) comes from the fact that $p' = g\zeta'$ at $z = 0$, which means that the surface is isopycnic (i.e. the density does not change on the surface). This way one is left with a classic Sturm–Liouville problem. Making the incompressibility approximation and assuming a uniform stratification in the reference state, namely $N = \bar{N} = \text{const.}$, it follows that

$$\omega^2 = \bar{N}^2 - mg \tan mH. \quad (34)$$

(Notice that to obtain m is necessary to fix a value of ω .) In the hydrostatic limit, $\omega^2 \ll \bar{N}^2$, it follows that the vertical normal modes result from

$$\tan mH = s/(mH), \quad (35)$$

where $s := \bar{N}^2 H/g$, which is a measure of the stratification, is such that $0 < s < \infty$ by static stability. In the ocean s is typically very small, so for $s \ll 1$ from (35) it follows that

$$m_i = \begin{cases} \bar{N}/\sqrt{gH} & \text{if } i = 0, \\ i\pi/H & \text{if } i = 1, 2, \dots \end{cases} \quad (36)$$

The first mode is called the *external* or *barotropic* mode; the rest of the modes are termed the *internal* or *baroclinic* modes, which are well separated from the latter in what length scale respects. More precisely, the *Rossby radii of deformation* are defined by $R_i := \bar{N}/(m_i |f_0|)$; for the barotropic mode $R_0 = \sqrt{gH}/|f_0|$ whereas for the baroclinic modes $R_i = \bar{N}H/(i\pi |f_0|) \equiv \sqrt{s}R_0/(i\pi) \ll R_0$. Finally, the *rigid lid approximation* consists of making $w' = 0$ at $z = 0$, which formally corresponds to take the limit $g \rightarrow \infty$ in (33b). This approximation filters out the barotropic mode since it leads to $\tan mH = 0$.

Problem 7 Demonstrate that $p' = g\zeta'$ at $z = 0$.

3. General Dispersion Relation

Upon eliminating c between (25) and (31) it follows that

$$\boxed{\frac{\mathbf{k}^2 + \beta k/\omega}{\omega^2 - f^2} = \frac{m^2}{N^2 - \omega^2} + c_s^{-2}}, \quad (37)$$

which is a fifth-order polynomial in ω that constitutes the general dispersion relation for linear ocean waves in an asymptotic WKB sense. This is the main result of this paper. Approximate roots of (37) are:

$$\text{acoustic} : \omega^2 = (\mathbf{k}^2 + m^2)c_s^2, \quad (38)$$

which holds for $\omega^2 \gg N^2$ (i.e. very high frequencies);

$$\text{Poincaré} : \omega^2 = \frac{\mathbf{k}^2 N^2 + m^2 f^2}{\mathbf{k}^2 + m^2}, \quad (39)$$

which follows upon taking the limit $c_s^{-2} \rightarrow 0$ and is valid for frequencies in the range $f^2 < \omega^2 < N^2$; and

$$\text{Rossby} : \omega = -\frac{\beta k}{\mathbf{k}^2 + (m^2/N^2)f^2}, \quad (40)$$

which also follows in the limit $c_s^{-2} \rightarrow 0$ but is valid for $\omega^2 \ll f^2$ (i.e. very low frequencies).

Problem 8 Demonstrate that the classical dispersion relations for Poincaré waves, $\omega^2 = f^2 + c^2 \mathbf{k}^2$, and surface gravity waves, $\omega^2 = g |\mathbf{k}| \tanh |\mathbf{k}| H$, are limiting cases of (39).

IV. DISCUSSION

The inviscid, unforced linearized equations of motion (7) have *five* prognostic equations for *five* independent dynamical fields. As a consequence, *five* is the number of waves sustained by (7) which satisfy the general dispersion relation (37) in an asymptotic WKB sense. In proper limits, two acoustic waves (AW), two Poincaré waves (PW), and one Rossby wave (RW) can be identified. The fact that the number of waves supported by the linearized dynamics equals the number of independent dynamical fields or prognostic equations, i.e. the degrees of freedom of (7), means that the waves constitute a complete set of solutions of the linearized dynamics (cf. Table I).

The number of possible eigensolutions can be reduced in approximations that eliminate some of the prognostic equations, or independent dynamical fields, of the system are performed. The Boussinesq approximation, which is very appropriate for the ocean, does not eliminate prognostic equations and has the virtue of reducing the mathematical complexity of the governing equations considerably. The incompressibility approximation, in turn, removes two degrees of freedom: the vertical velocity is diagnosed by the horizontal velocity,

$$\partial_z w' = -\nabla \cdot \mathbf{u}', \quad (41)$$

and the latter along with the density diagnose the pressure field through the three-dimensional Poisson equation

$$(\nabla^2 + \partial_{zz})p' = -\nabla \cdot f \hat{\mathbf{z}} \times \mathbf{u}' - \partial_z(N^2 \zeta'). \quad (42)$$

$$\left. \begin{array}{l} \partial_t \zeta' = \dots \\ \partial_t p' = \dots \\ \partial_t \mathbf{u}' = \dots \\ \partial_t w' = \dots \end{array} \right\} : 5 \text{ independent fields} \leftrightarrow 5 \text{ waves} : \begin{cases} 2 \text{ AW} \\ 2 \text{ PW} \\ 1 \text{ RW} \end{cases}$$

$$\text{incompressibility} \downarrow \begin{cases} \partial_z w' = -\nabla \cdot \mathbf{u}' \\ \Delta p' = -\nabla \cdot f \hat{\mathbf{z}} \times \mathbf{u}' - \partial_z(N^2 \zeta') \end{cases}$$

$$\left. \begin{array}{l} \partial_t \zeta' = \dots \\ \partial_t \mathbf{u}' = \dots \end{array} \right\} : 3 \text{ independent fields} \leftrightarrow 3 \text{ waves} : \begin{cases} 2 \text{ PW} \\ 1 \text{ RW} \end{cases}$$

$$\text{quasigeostrophy} \downarrow \partial_z \mathbf{u}' = \frac{N^2}{f_0} \hat{\mathbf{z}} \times \nabla \zeta'$$

$$\partial_t q' = \dots : 1 \text{ independent field} \leftrightarrow 1 \text{ wave} : 1 \text{ RW}$$

TABLE I: Reduction of independent fields (and, hence, prognostic equations) by the incompressibility and quasigeostrophic approximations. Here, AW, PW, and RW stand for acoustic waves, Poincaré waves, and Rossby wave, respectively; $\Delta := \nabla^2 + \partial_{zz}$ is the three-dimensional Laplacian; and $q' := f + \nabla^2 p'/f_0 + \partial_z(f_0 N^{-2} \partial_z p')$ is the so-called quasigeostrophic potential vorticity.

As a consequence, the two AW are filtered out and one is left with the two PW and the RW. With these two approximations the Euler equations (7) reduces to the so called *Euler-Boussinesq equations*. Performing in addition the hydrostatic approximation, which corresponds to neglecting $\partial_t w'$ in Newton's vertical equation, does not amount to a reduction of degrees of freedom because the vertical velocity is already diagnosed by the horizontal velocity. In this case, the density field diagnoses the pressure field through

$$\partial_z p' = -N^2 \zeta'. \quad (43)$$

With this approximation (which implies that of incompressibility) and the Boussinesq approximation, system (7) reduces to what is known in geophysical fluid dynamics as the *primitive equations*. Finally, one approximation that eliminates independent fields is the *quasigeostrophic approximation*, which is often used to study low frequency motions in the ocean, and the Earth and planetary atmospheres. In this approximation the density diagnoses the horizontal velocity through the "thermal wind balance,"

$$\partial_z \mathbf{u}' = \frac{N^2}{f_0} \hat{\mathbf{z}} \times \nabla \zeta', \quad (44)$$

thereby removing two degrees of freedom and leaving only one RW.

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APPENDIX A: SOLUTIONS TO SOME OF THE PROBLEMS

Problem 1 To describe the dynamics in a noninertial reference frame such as one tied to the rotating Earth, two forces must be included: the Coriolis and centrifugal forces. However, Laplace¹ showed that if the upward coordinate z is chosen not to lie in the direction of the gravitational attraction, but rather to be slightly tilted toward the nearest pole, the centrifugal and gravitational forces can be made to balance one another in a horizontal plane (cf. also Ref. 8). With this choice the Coriolis force is the only one needed to describe the dynamics. Notice that the absence of the centrifugal force in a system fixed to the Earth is what actually makes rotation effects real: they cannot be removed by a change of coordinates.

Problem 2 In the absence of diffusive processes, the isopycnal $z = \zeta$ is a material surface, i.e. $[\mathbf{u} + \hat{\mathbf{z}} w - (\mathbf{u}_\zeta + \hat{\mathbf{z}} w_\zeta)] \cdot [\nabla\zeta - \hat{\mathbf{z}}(1 - \partial_z\zeta)] = 0$. Here, $\mathbf{u}_\zeta + \hat{\mathbf{z}} w_\zeta$ denotes the velocity of *some* point on the surface [the velocity of a surface is not defined and it only makes sense to speak of the velocity in a given direction, e.g. the normal direction, in whose case it is $\hat{\mathbf{z}}(1 - \partial_z\zeta) - \nabla\zeta$]. From the trivial relation $z - \zeta = 0$ then it follows that $(\mathbf{u}_\zeta + \hat{\mathbf{z}} w_\zeta) \cdot [\nabla\zeta - \hat{\mathbf{z}}(1 - \partial_z\zeta)] = -\partial_t\zeta$ and, hence, $D\zeta = w$ at $z = \zeta$.

Problem 3 To perform the linearization of the equations of motion, we write

$$\begin{aligned} (\mathbf{u}, w, \zeta) &= (\mathbf{u}', w', \zeta') + \dots, \\ (p, \rho) &= (\rho_r, p_r) + (\rho', p') + \dots, \\ O &: 1 \quad a \quad a^2 \end{aligned} \quad (\text{A1})$$

where a is an infinitesimal amplitude. The $O(a)$

continuity equation (7b) readily follows upon noticing that, up to $O(a)$, $c_s^{-2}Dp = c_s^{-2}(\partial_t p' - \rho_r g w')$. Up to $O(a)$, $D\rho - c_s^{-2}Dp = \partial_t \rho' - g^{-1} \rho_r N^2 w' - c_s^{-2} \partial_t p'$ and $D\zeta - w = \partial_t \zeta' - w'$. Then from the relationships $D\rho - c_s^{-2}Dp = 0$ and $D\zeta - w = 0$ it follows that $\rho' = c_s^{-2} p' + g^{-1} \rho_r N^2 \zeta'$. Bearing in mind the latter relation and the fact that $dp_r/dz = -g\rho_r$, the $O(a)$ vertical Newton's equation (7d) then follows.

Problem 4 For the ocean $c_s \sim 1500 \text{ m s}^{-1} \gg \sqrt{gH} \sim 200 \text{ m s}^{-1}$; by contrast, for the atmosphere $c_s \sim 350 \text{ m s}^{-1} \sim \sqrt{gH}$ with $H \sim 12 \text{ km}$, which is the typical height of the troposphere.

Problem 7 At the surface $z = \eta$ it is $w = \partial_t \eta + \mathbf{u} \cdot \nabla \eta$ and $p = 0$ (here, p is a kinematic pressure, i.e. divided by a constant reference density $\bar{\rho}$). Writing $\eta = \eta' + O(a^2)$ and Taylor expanding about $z = 0$ it follows, on one hand,

$$w' + \eta' \partial_z w' + O(a^3) = \partial_t \eta' + \mathbf{u}' \cdot \nabla \eta' + O(a^3) \quad (\text{A2})$$

at $z = 0$, and, on the other hand,

$$p_r + (p_r + dp_r/dz)\eta' + \eta' \partial_z p' + O(a^3) = 0 \quad (\text{A3})$$

at $z = 0$. From (A2) it follows, to the lowest order, $w' = \partial_t \eta'$ at $z = 0$. Since $w' = \partial_t \zeta'$, for a wave (i.e. $\partial_t \neq 0$) then it follows that $\eta' = \zeta'$ at $z = 0$. Taking into account the latter and choosing $\bar{\rho} = \rho_r(0)$, from (A3) it follows, to the lowest order, $p' = g\eta' \equiv g\zeta'$ at $z = 0$ since $p_r = 0$ and $dp_r/dz = -g\rho_r/\bar{\rho} \equiv -g$ at $z = 0$.

Problem 8 The classical dispersion relation for Poincaré waves corresponds to the hydrostatic limit, which requires $m^2 \gg \mathbf{k}^2$ (i.e. that the vertical length scales be shorter than the horizontal length scales). Under this conditions, $\omega^2 = f^2 + \mathbf{k}^2 N^2/m^2 = f^2 + c^2 \mathbf{k}^2$. To obtain the dispersion relation for surface gravity waves one needs to take into account boundary conditions (33): making $N^2 \equiv 0 \equiv f^2$ it follows, on one hand, that $m^2 = -\mathbf{k}^2$, and, on other, that $\omega^2 = -mg \tanh mH$. The dispersion relation $\omega^2 = g|\mathbf{k}| \tanh |\mathbf{k}| H$ then readily follows.

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