# Linear Waves and Baroclinic Instability in an Inhomogeneous-Layer Primitive-Equation Ocean Model

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## ABSTRACT

We consider a multilayer generalization of Ripa's model. In addition to vary arbitrarily in horizontal position and time, the horizontal velocity and buoyancy fields are allowed to vary linearly with depth within each layer of the model. As a base test for the validity of the model we consider how well linear waves and baroclinic instability are represented in the model. We find very accurate results with the inclusion of only a small number of layers.

## 1 The Generalized Model

In one possible generalization[1] of Ripa's model,[3] a stack of $n$ inhomogeneous layers, each of thickness $h_i(\mathbf{x},t)$, is considered in a reduced-gravity setting. The $i$th-layer horizontal velocity and buoyancy are written as

$$\mathbf{u}_i(\mathbf{x},\sigma,t) = \bar{\mathbf{u}}_i(\mathbf{x},t) + \sigma\mathbf{u}_i^\sigma(\mathbf{x},t), \quad \vartheta_i(\mathbf{x},\sigma,t) = \bar{\vartheta}_i(\mathbf{x},t) + \sigma\vartheta_i^\sigma(\mathbf{x},t). \tag{1}$$

Here, the overbar stands for vertical average within the $i$th layer, and $\sigma$ is a scaled vertical coordinate which varies linearly from $-1$ at the base of the $i$th layer to $+1$ at the top of the $i$th layer. The model equations, which follow upon replacing (1) in the continuously and arbitrarily stratified (i.e. exact) primitive equations (i.e. rotating hydrostatic incompressible Euler–Boussinesq equations), then take form

$$\partial_t h_i + \nabla\cdot h_i\bar{\mathbf{u}}_i = 0, \tag{2a}$$

$$\overline{\mathrm{D}_t\vartheta_i} = 0, \quad (\mathrm{D}_t\vartheta_i)^\sigma = 0, \tag{2b,c}$$

$$\overline{\mathrm{D}_t\mathbf{u}_i} + f\hat{\mathbf{z}}\times\bar{\mathbf{u}}_i + \overline{\nabla p}_i = \mathbf{0}, \quad (\mathrm{D}_t\mathbf{u}_i)^\sigma + f\hat{\mathbf{z}}\times\mathbf{u}_i^\sigma + (\nabla p_i)^\sigma = \mathbf{0}. \tag{2d,e}$$

Some of the properties enjoyed by the model are: (1) the model can represent explicitly within each layer the thermal-wind balance which dominates at low frequency; (2) volume, mass, buoyancy variance, energy, and momentum are preserved by the dynamics; (3) the model equations can be cast as a generalized Hamiltonian system; and (4) thermodynamic processes (e.g. due to heat and freshwater inputs across the ocean surface, localized vertical mixing events, etc.) can be incorporated in the model.

## 2 Notation

Consistent with the notation used in Ref. 3, a model with $n$ inhomogeneous layers (ILs) is denoted $n$-IL$^m$ , where the superscript indicates the amount of vertical variation allowed in the sense of the degree of a polynomial in depth. The exact model is denoted IL$^\infty$ and a model with $n$ homogeneous layers (HLs) is denoted $n$-HL.

## 3 Waves

System (2), linearized with respect to a steady solution of (2) with no currents (i.e. *reference state*), can be shown to sustain the usual geophysical waves in $2n$ vertical normal modes. In this work we concentrate on how well these modes are represented by considering the phase speed of internal long gravity waves in a reference state characterized by the stratification parameter $S := \frac{1}{2}N_\mathrm{r}^2 H_\mathrm{r}/g_\mathrm{r}$, which must be such that $0 < S < 1$.[2,3] Here, $N_\mathrm{r}$ is the Brunt–Väisälä frequency, $H_\mathrm{r}$ is the total thickness of the active fluid layer, and $g_\mathrm{r}$ denotes the vertically averaged buoyancy. The reference buoyancy varies linearly from $g_\mathrm{r}(1+S)$ at the top of the active layer to $g_\mathrm{r}(1-S)$ at the base. Phase speeds, as a function of $S$, are depicted in Fig. 1 for several vertical modes as predicted by the exact model, and the present and two additional layer models.

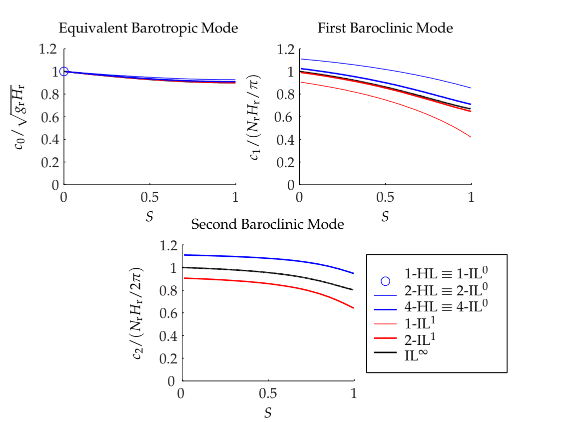


**Figure 1.** Phase speed for internal long gravity waves as a function of reference state stratification parameter.

## 4 Baroclinic Instability

We now consider a steady solution of (2) with a parallel current lying on an infinite channel of the $f$ plane (i.e. a *basic state*). For future reference we define the lengthscale $R := \sqrt{g_\mathrm{r}H_\mathrm{r}}/|f|$ and let Ro be the Rossby number.

When Ro $\to 0$, the basic velocity can be chosen to vary linearly from $\bar{U} + U^\sigma$ at the top of the active layer to $\bar{U} - U^\sigma$ at the base of the active layer. According to the thermal-wind relation, the basic buoyancy varies from $g_\mathrm{r}(1 - 2fU^\sigma y/H_\mathrm{r} - S)$ at the top of the active layer to $g_\mathrm{r}(1 - 2fU^\sigma y/H_\mathrm{r} + S)$ at the base ($y$ is across channel). A nonzero velocity at the base implies that the latter has a linear $y$-slope $g_\mathrm{r}^{-1}f\,(U^\sigma - \bar{U})\,/\,(1-S)$. Infinitesimal low-frequency, i.e. $O(\mathrm{Ro})$, normal-mode perturbations are considered to this state with $S \to 0$ and $|\bar{U}/U^\sigma| \ll O(S^{-1})$. Top panel in Fig. 2 shows minimum along-channel wavenumber, $k$, for instability as a function of $\bar{U}/U^\sigma$ with $kR = O(1)$ (free-boundary baroclinic instability[2]). Bottom panel in Fig. 2 depicts, as a function of $k$, growth rate of the most unstable perturbation with $kR = O(S^{-1/2})$ (classical baroclinic instability). Results are showed for the exact model, and the present and two additional layer models.

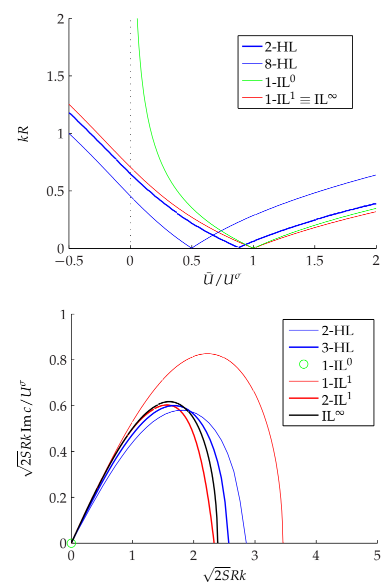


**Figure 2.** Minimum wavenumber for free-boundary baroclinic instability (top) and maximum growth rate of classical baroclinic instability (bottom).

When Ro $\lesssim 1$, for the above state to be a basic state of (2) it must have $\bar{U} \equiv U^\sigma =: U$. Taking $R$ as the relevant lengthscale, the Richardson number Ri $= \frac{1}{2}S\mathrm{Ro}^{-1}$. Figure 3 shows, as a function of Ri, maximum growth rate (left) and corresponding wavenumber (right) of an infinitesimal non-geostrophic normal-mode perturbation superimposed to this basic state. The present layer solutions are confronted with those of the exact model in the figure.

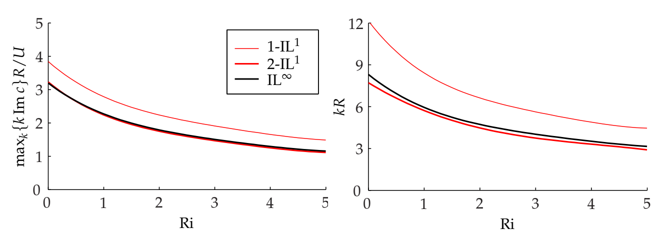


**Figure 3.** Maximum growth rate (left) and corresponding wavenumber (right) as a function of the Richardson number in the ageostrophic regime.

## 5 Conclusions

We examined the performance of a layer model featuring vertical shear and stratification within each layer in two aspects of ocean dynamics, namely linear waves and baroclinic instability. Consideration of a few layers is found enough to describe accurately these aspects. We expect that a model configuration involving a small number of layers will set the basis for a quite accurate—and still physically insightful—ocean model.

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[1] F. J. Beron-Vera, J. Fluid Mech., in revision (2004, e-Print arXiv:physics/0312083).
[2] F. J. Beron-Vera and P. Ripa, J. Fluid Mech. **352**, 245 (1997).
[3] P. Ripa, J. Fluid Mech. **303**, 169 (1995).