CSC688/MTH686: Scientific Computation
The Sturm–Liouville Eigenproblem

November 9, 2006

Let $x \in [a, b] \in \mathbb{R}$. Consider then the second-order ODE

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) + (\beta(x) + \lambda\gamma(x))u = 0, \tag{1}$$

where λ is an unspecified parameter. We call the above a *regular Sturm–Liouville eigenproblem* if $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are all continuous, $\alpha(x)$ and $\gamma(x) > 0$, and $u(x)$ satisfies the boundary conditions

$$\begin{aligned} a_1u(a) + a_2u'(a) &= 0, & (2)\\ b_1u(b) + b_2u'(b) &= 0. & (3)\end{aligned}$$

It can be shown that the regular Sturm–Liouville eigenproblem has the following properties:

1. All eigenvalues are real.
2. There exists an infinite number of numerable eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots < \lambda_\infty. \tag{4}$$

3. Associated with each eigenvalue λ_n there is an eigenfunction $u_n(x)$ with exactly $n - 1$ zeros in $[a, b]$.
4. The eigenfunctions $\{u_n(x)\}$ form a complete basis, i.e., they satisfiy

$$\sum_n u_n(x)u_n(y) = \delta(x - y). \tag{5}$$

   Consequently, any (piecewise continuous) function $f(x)$ can be expressed a linear combination of the elements of that basis:

$$f(x) = \sum_n c_n u_n(x). \tag{6}$$

5. The eigenfunctions $\{u_n(x)\}$ constitute an orthogonal set with respect to the weight function $\gamma(x)$, i.e.,

$$\int_a^b u_n(x)u_m(x)\gamma(x) = 0 \text{ if } n \neq m. \tag{7}$$