CSC688/MTH686: Scientific Computation
Equation Solving

October 10, 2006

1. Let $f : \mathbb{R}^n \to \mathbb{R}^n;\ x \mapsto f(x)$ be of class C^1. Consider $g : \mathbb{R}^n \to \mathbb{R}^n$ defined by
$$g(x) := x - \nabla f(x)^{-1} f(x). \tag{1}$$
Verify that $|\det \nabla g(r)| < 1$ where $f(r) = 0$.

2. Let $f : \mathbb{C} \to \mathbb{C};\ z \mapsto f(z)$ and consider the problem of solving $f(z) = 0$ using Newton's method. Demonstrate that if $f(z)$ is analytic then Newton's recursive formula is given by
$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \tag{2}$$
Hint: Write $f(z) = g(x, y, t) + \mathrm{i}h(x, y, t)$. Then note that $f'(z) = \frac{1}{2}(g_x + h_y) + \frac{\mathrm{i}}{2}(h_x - g_y)$ and furthermore that if $f(z)$ is analytic then $f'(z^*) = 0$.

3. Suppose that the locations of planets M and E at time t are given by
$$x_{\mathrm{M}} = -11.9084 + 57.9117 \cos \frac{2\pi t}{87.97}, \tag{3}$$
$$y_{\mathrm{M}} = 56.6741 \sin \frac{2\pi t}{87.97}, \tag{4}$$
and
$$x_{\mathrm{E}} = -2.4987 + 149.6041 \cos \frac{2\pi t}{365.25}, \tag{5}$$
$$y_{\mathrm{E}} = 149.5832 \sin \frac{2\pi t}{365.25}. \tag{6}$$
These are crude impersonifications of Mercury and Earth. Both orbits are elliptical with one focus, the Sun, located at $(x, y) = (0, 0)$. To an observer on E, M is in *conjunction* if it is located on the Sun-to-Earth line segment. Clearly, there is a conjunction at $t = 0$. The proposed problem is to compute the time of the next 10 conjunctions. **Recommendation:** Use FZERO.M.

4. In the analysis of vertical profiling oceanographic data it is often desirable to transform the depth dependent measurements into vertical wavenumber

space. However, the vertical structure of oceanic variables is strongly depth dependent; viz., with larger gradients (smaller length scales) near the surface compared to deeper parts of the water column. This may be characterized by the squared Brunt–Väisälä frequency profile, $N^2(z)$. Instead of the the sines and cosines of Fourier analysis (solutions of $F''(t)+\omega^2 F(t) = 0$) the natural functions to transform the data are the solutions of

$$F''(z) + c^{-2} N^2(z) F(z) = 0. \tag{7}$$

Vertical displacements are projected onto the $F(z)$—density perturbations onto $N^2(z)F(z)$—while horizontal velocities and pressure deviations onto their derivatives $F'(z)$. A discrete set of eigensolutions $(F_n(z), c_n)$, $n = 0, 1, 2, \cdots$, is obtained by adding two boundary conditions to (7). For instance, if $N^2(z)$ is known from the surface, $z = 0$, to the (flat) bottom, $z = -H$, the solutions of (7) subject to

$$F(-H) = 0, \tag{8}$$

$$F'(0) = gc^{-2} F(0), \tag{9}$$

where g is the acceleration of gravity, are the dynamical normal modes corresponding to a given $N^2(z)$. The set $\{F_n(z)\}$ constitutes a complete orthogonal basis. For typical oceanic conditions ($\int_{-H}^{0} N^2(z)\,\mathrm{d}z \ll g$), the first eigensolution ($n = 0$) is the *external* or *barotropic* mode (for which $c_0^2 \sim gH$ and $F(z) \sim z + H$) and the others are the *internal* or *baroclinic* modes (for which $c_n^2 \ll c_0^2$, $n = 1, 2, \cdots$). The proposed problems are the following.

(a) Demonstrate that $\{F(z)\}$ constitute a complete basis of orthogonal functions.

(b) Consider $N = \bar{N} = \text{const}$ and demonstrate that

$$\tan \frac{\bar{N} H}{c_n} = \frac{\bar{N} c_n}{g}, \tag{10}$$

which is a transcendental equation for c_n.

(c) Consider then the typical oceanic values $\bar{N} = 10^{-5}$ s^{-1}, $H = 4 \times 10^3$ m, and $g = 9.8$ m s^{-2}, and solve numerically (10) for the first 10 modes. **Hint:** When $\bar{N}^2 H/g \ll 1$ it is easy to see that $c_0 = \sqrt{gH}$ and $c_n = \bar{N}H/n\pi$, $n = 1, 2, \cdots$. **Recommendation:** Use FZERO.M.