Numerical tests on vortex polygons
Zhenduo Zhu
Florida International University

1. Introduction

Point vortex, first introduced by Helmholtz in 1958, is a classic model in two-dimensional incompressible fluid dynamics. The solutions to the equations of various configurations of point vortices have provided many insightful findings in various physical fields involved with vortex dynamics, such as atomic structure, large scale weather patterns and superconductors, etc (Aref 1988). One kind of such configurations of great interest is vortex polygon, in which N identical vortices sit on the corners of a regular polygon (Aref 1983, 1995). Because of its symmetry, vortex polygon rotates as a whole without change of shape, and it is often referred as the best known equilibrium for identical vortices. The problem of vortex polygon was first studied by J.J. Thomson in his essay for the Adams Prize of 1882 (Thomson 1883). He proved that this configuration is linearly stable for N<=6 and unstable for N>=7, to infinitesimal perturbations. Since then a lot of work has been done on this problem. Morikawa et al. (1971) and Aref (1995) found that the heptagon (N=7) is actually neutrally stable in linear theory. As a natural extension of regular vortex polygon, body-centered (N+1) vortex polygon is also well studied. Mertz (1978) showed that a strong enough central vortex can stabilize a regular vortex polygon. Cabral et al. (1999) added a upper limit to the strength of the central vortex in question. Except the body-centered vortex polygons, Campell et al. (1978, 1979) found some other stable equilibrium configurations for identical vortices by placing them on concentric circles. Aref (1995) established a connection between vortex polygon and vortices in an infinite row by showing that the stability problem for both configurations can be solved by the same eigenvalue problem for a certain symmetric matrix.

Most of these findings on vortex polygons were obtained from purely theoretical studies. As is well known, numerical simulations can capture the complicated details throughout the process of a certain physical phenomenon which are often missed by the theoretical analyses, thus is a necessary complement to theoretical studies. By running several numerical tests with a two-dimensional model, this study roughly investigates three details on vortex polygons: 1) How big a perturbation can break up a stable configuration? In other words, what is generally the largest allowed amplitude of the "infinitesimal perturbations" aforementioned in a numerical model? ; 2) Will a unstable configuration evolve into a completely chaotic state, or a state of certain pattern? 3) If Cabral et al. is right about the strength of the central vortex to stabilize a regular vortex polygon, what will happen to a body-centered polygon with a overly strong central vortex?

2. Model description

The numerical model used in this study is a two-dimensional incompressible flow model based on two-dimensional vortex methods (Chorin 1993, Gustafson 1991). It is also fully-nonlinear, inviscid and unbounded. The model uses Mollified kernal to represent the radial distribution of velocity and vorticity (Fig. 1) of a vortex blob. The kernel smoothes out the singularity of velocity near vortex center. This model does not include any deformation process hence rules out any merger of blobs. In essence, the blobs in this model move in the same manner as the point vortices, except for the effect of the blob size on the velocity field. The general settings of the model for this study are listed in table 1.

<table>
<thead>
<tr>
<th>Spatial Resolution (m)</th>
<th>Time Step (s)</th>
<th>Blob Strength (m$^2$s$^{-1}$)</th>
<th>Blob Size (m)</th>
<th>Polygon Radius (m)</th>
<th>Simulation Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>1.0</td>
<td>0.1</td>
<td>1.0</td>
<td>$10^6$</td>
</tr>
</tbody>
</table>

Table 1. Model settings. Some tests use different time step and will be specified later.

Figure 1. Radial profiles of velocity (left) and vorticity (right) of Mollified kernal.
3. Tests and results

3.1 Amplitude of "infinitesimal perturbations"

From theoretical stability analysis, a stable/unstable state subject to infinitesimal perturbations must suppress/amplify the perturbations. If the perturbations are not small enough, they can break up the stable state. Therefore, to test stability using numerical models, we first need to know generally the largest allowed amplitude of these infinitesimal perturbations. In this section, the amplitude of perturbations (to the vortex position) is tested for 6-gon and 7-gon. These two configurations are chosen because they have the weakest stability among all the stable polygons, hence have strongest restriction in the amplitude of perturbations. The amplitude is nondimensionalized by the arc spacing of vortices, i.e.,

\[ A_n = \frac{AN}{2\pi R} \]

where \( A_n \) and \( A \) are the dimensionless and the dimensional amplitude respectively, \( N \) is the number of vortices and \( R \) is the radius of the polygon. The perturbation is added to one vortex in the radial or azimuthal direction (Fig. 2). The tested \( A_n \)'s are 0.001, 0.01, 0.1 and 0.2. The evolution of the perturbation is represented by the maximum nondimensionalized deviation of vortices from the initial polygon circle.

For 6-gon (Fig. 3), which is a stable configuration, all tests for \( A_n \) of 0.001, 0.01 and 0.1 show suppression of the perturbations. Tests for \( A_n \) of 0.2 show that the radial perturbation is suppressed, while the azimuthal perturbation is not. In the latter test, the 6-gon breaks up by moving on vortex towards the center and evolves into a 5+1-gon quasi-stable configuration. For 7-gon (Fig. 4), which is a neutral stable configuration, only the perturbations of \( A_n \) of 0.001 and 0.01 are not amplified. Therefore, the safest nondimensionalized amplitude of "infinitesimal perturbations" for numerical stability tests is generally smaller than or of the order of 0.01.

3.2 Evolution of unstable vortex polygons

In this section, the evolution of 8-gon, 9-gon and 10-gon without initial perturbation is examined. In all these three tests, the vortex polygons quickly break up due to the perturbation induced by the truncation error of the numerical model, and the distribution of vortices appears to be completely chaotic after a long-term simulation. Fig. 5 shows a snapshot of the distribution of 8-gon vortices at \( T = 9.9 \times 10^5 \) seconds. All the vortices are confined within a patch of radius about 1.5 around the origin, which is an expected result of the conservation of vorticity centroid and dispersion in two-dimensional vortex dynamics.

![Figure 2. Example of radial (left) and azimuthal (right) perturbation. The vortex marked by arrow is the one being perturbed.](image-url)
Figure 3. Evolution of perturbation (left) of $n_A=0.1$ (up) and 0.2 (down) and snapshot of vortices for $n_A=0.2$ at $T=10^6$ s (right) from 6-gon tests. The evolution of the perturbation is represented by the maximum non-dimensionalized deviation of vortices from the initial polygon circle.

Figure 4. Evolution of perturbation of $n_A=0.01$ (left) and 0.1 (right) from 7-gon tests.
3.3 Strength of central vortex

In this section, the strength of the central vortex is tested for 8-gon. The ratio of the strength of central vortex to the strength of polygon vortices is denoted as $P$. According to Cabral et.al (1999), to stabilize the 8-gon, $P$ needs to be within $(0.5, 12.25)$. Therefore, $P$ of 1, 5, 10 and 12.5 are tested. Because with stronger central vortex, the angular velocity of vortices is higher, the time step needs to be adjusted to satisfy the stability conditions for the numerical difference scheme. The tests for $P$ of 5, 10 and 12.5 use the time resolution of 0.01. Consistent with Cabral et al. (1999), the 8+1-gon is stable in the tests for $P$ of 1, 5 and 10 (not shown), but is unstable in test for $P$ of 12.5 (Fig. 6). Examination of the evolution of the 8+1-gon in the last test shows that the 8+1-gon starts to break around 500 seconds, then gradually evolves into a half-moon-like configuration, in which the 8-gon vortices move closer and rotate together with no change in angular velocity along the circle on which they initially sit, and the central vortex also rotates along an inner circle to make the vorticity centroid and dispersion conserved. An interesting finding is that this half-moon-like configuration appears to be completely stable after the simulation reaches $10^6$ seconds.

4. Summary

Using a two-dimensional vortex model, this study performed several numerical tests on vortex polygon. The tests are focused on the amplitude of perturbations appropriate to numerical stability analysis, the evolution of unstable vortex polygons, and the strength of central vortex in body-centered polygons. The appropriate perturbation for numerical stability analysis is found to be of nondimensionalized amplitude smaller than or of the order of 0.01. The unstable polygon breaks up due to numerical truncation error. Long-term simulation shows that it evolves into a chaotic state but is always confined within a certain area around the its center. Long-term simulation of a body-centered polygon with an overly strong center vortex shows that its evolution ends up with a half-moon-like stable configuration.
5. References


