Empirical Orthogonal Function Analysis of Ocean Surface Currents Using Complex and Real-Vector Methods *

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(Manuscript received 8 April 1997, in final form 24 September 1997)

ABSTRACT

Empirical orthogonal function (EOF) analysis has been widely used in meteorology and oceanography to extract dominant modes of behavior in scalar and vector datasets. For analysis of two-dimensional vector fields, such as surface winds or currents, use of the complex EOF method has become widespread. In the present paper, this method is compared with a real-vector EOF method that apparently has previously been unused for current or wind fields in oceanography or meteorology. It is shown that these two methods differ primarily with respect to the concept of optimal representation. Further, the real-vector analysis can easily be extended to three-dimensional vector fields, whereas the complex method cannot. To illustrate the differences between approaches, both methods are applied to Ocean Surface Current Radar data collected off Cape Hatteras, North Carolina, in June and July 1993. For this dataset, while the complex analysis “converges” in fewer modes, the real analysis is better able to isolate flows with wide cross-shelf structures such as tides.

1. Introduction

The empirical orthogonal eigenfunction (EOF) method, first introduced to geophysics by Lorenz (1956), has subsequently been used to enable analysis of data with complex spatial/temporal structures. Using this method, the most efficient decomposition of the data into representative modes is determined by empirically finding the eigenfunctions that best describe the information. It can be proven that the EOF method describes the data in the most compact form in a sense to be described below. The EOF eigenmodes can be ordered in terms of the percentage of the total variance to be described by each mode and, in addition, the modes are statistically uncorrelated with one another (Sirovich 1987).

The method is useful in this regard for two reasons. First, retention of only the first few modes may contain a significant portion of the total variance, leading to potentially significant storage savings if not all the variance is required. Additionally, each mode contains phenomena with differing spatial/temporal scales and thus can be isolated. They can then be associated with physical processes being studied. The method (also called principal component analysis or Karhunen–Loeve analysis) is empirical because the data are used to find their own optimum decomposition with no a priori assumptions on either spatial or temporal behavior. This optimization is found by formulating an eigenvalue problem involving the two-point spatial covariance matrix.

In meteorology and oceanography, EOF analysis has been used to infer the driving forces and spatial/temporal scales behind the variability of sea surface temperature (Kutzbach 1967), winds (Hardy 1977; Legler 1983), currents (Kundu and Allen 1976; Klinck 1985; Prandle and Matthews 1990; Ng 1993), and beach profile evolution (Winant et al. 1975). Prior to the study of Kundu and Allen (1976), the individual components of ocean current fields were often analyzed separately; independent EOF analyses were performed on the scalar components \((u, v)\) of a current field and the results an-
alyzed as decoupled scalars. Kundu and Allen (1976) employed the complex EOF analysis,\(^1\) which essentially forms complex scalars from the \(u, v\) components of the vector field. This method has since been used almost exclusively in oceanography and meteorology. Another EOF analysis method, referred to here as real-vector EOF analysis, concerns a direct treatment of the vector nature of the problem and will be described in detail in section 2b. There do not appear to be any references to the application of this method for current or velocity fields in oceanography or meteorology.

In this study two separate EOF analyses of ocean surface current data are investigated. The dataset used is unique in that high spatial resolution was achieved, offering an intriguing opportunity to discern flows with high spatial variability. Both the complex EOF and real-vector EOF analyses are used on a common dataset to compare the two methods and investigate the differences in the results. We will also note mathematical differences between the methods, as well as describe how the results of the two methods retain the physical properties of the analyzed flow.

2. Complex and real-vector EOF analysis

a. Complex EOF analysis for current fields

The complex EOF technique has been described in numerous studies (Kundu and Allen 1976; Klinck 1985; Prandle and Matthews 1990). The objective of the EOF analysis is to optimally describe the surface current field \(u(x, t) = (u, v, w)\), where \(u, v) = (x, v)\) are east–west and north–south velocity components, respectively, and \(x = (x, y)\) represents the corresponding coordinate system. In the complex analysis, the current field \(u\) is transformed into a complex scalar, \(\mathbf{U}(x, t) = u(x, t) + jv(x, t)\), where \(j = (-1)^{1/2}\). We now seek a complex scalar field \(\Phi(x)\) that best represents \(U(x, t)\) in the mean square sense.

The inner product \((bU, \Phi(x))\) is defined by

\[
(U, \Phi) = \int U\Phi^* \, dx. \tag{2.1}
\]

In addition, the quantity \(\lambda\), a measure of the energy or variance associated with a given mode, is defined by

\[
\lambda = \langle |(U, \Phi)|^2 \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T |(U, \Phi)|^2 \, dt, \tag{2.2}
\]

where \(\langle \rangle\) represents ensemble averaging, and the equivalence of temporal and ensemble averaging implied here is valid for ergodic processes. More precisely, the function(s) \(\Phi\) that maximize \(\lambda\) under the constraint \((\Phi, \Phi) = 1\) are sought. A variational analysis of the problem immediately leads to the integral equation given by

\[
\int R(x, x')\Phi_l(x') \, dx' = \lambda_l\Phi_l(x), \tag{2.3}
\]

where \(R(x, x') = \langle U(x)U^*(x') \rangle\). Then the eigenvalue problem given by (2.3) leads to a complete set of orthonormal eigenfunctions, \(\Phi_l\), each corresponding to an eigenvalue, \(\lambda_l\), which can be directly associated with the variance or energy in each eigenmode. Thus, the complex scalar field \(U(x, t)\) can be expanded in terms of \(\Phi_l\) by

\[
U(x, t) = \sum_{l} a_l(t)\Phi_l(x), \tag{2.4}
\]

where the series converges optimally in the mean square sense and the time-dependent coefficients are given by \(a_l(t) = \langle U, \Phi_l \rangle\). Note that these coefficients are complex and are typically represented by their amplitude and phase. Additionally, the complex eigenfunctions, \(\Phi_l\), can be represented by \(\Phi_l = \Phi_l^* + j\Phi_l^\perp\), where the subscripts \(u\) and \(v\) refer to the corresponding current components.

In the following, assuming that current data at \(M\) spatial locations defined by \(x_m(n = 1, M)\) and at \(N\) instants in time, \(t_m(n = 1, N)\) are available, Eq. (2.3) can be discretized as

\[
\sum_{m=1}^{M} R(x_m, x_n)\Phi_l(x_n) = \lambda_l\Phi_l(x_m), \tag{2.5}
\]

and the covariance matrix is given by

\[
R(x_m, x_n) = \frac{1}{N} \sum_{n=1}^{N} U(x_m, t_n)U^*(x_n, t_n). \tag{2.6}
\]

The eigenvalues computed from (2.5) are real and non-negative. In addition, if we define the total energy, \(E\), by \(E = \text{Tr}[R(x_m, x_m)]\), then

\[
E = \sum_{m=1}^{M} \lambda_m, \tag{2.7}
\]

so that we may interpret each mode \(\Phi_l\) as contributing a certain percentage of the total energy or variance in the current flow field. We have implied in (2.5)–(2.7) that there are \(M\) distinct eigenfunctions, which follows from the fact that \(R\) is an \(M \times M\) complex Hermitian matrix.

b. Real-vector analysis for current fields

Though the real-vector EOF analysis method has been used extensively in the study of turbulence (e.g., Lumley 1967; Bakewell and Lumley 1967; Payne and Lumley 1967; Sirovich 1987; Sirovich et al. 1991; Holmes et al. 1996; Webber et al. 1997), we can find no reference to its use in the meteorological or oceanographic lit-

\(^1\) The term complex EOF analysis has also been used to refer to analysis of Fourier transforms of scalar data in an attempt to determine phase speeds (e.g., Merrill and Guza 1990); this method should not be confused with the method described in this study.
erature. In this approach an inner product \((\mathbf{u}, \mathbf{\Psi})\) is defined by
\[
(\mathbf{u}, \mathbf{\Psi}) = \int u_j \mathbf{\Psi}_j \, d\mathbf{x}, \quad (j = 1, 2),
\] (2.8)
where \(u_j = (u_1, u_2), u_1 = u, \) and \(u_2 = v, \) and repeated indices imply summation. The function(s) \(\mathbf{\Psi}\) that maximize \(\lambda\) are sought from the expressions given by
\[
\lambda = \langle (\mathbf{u}, \mathbf{\Psi})^2 \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T (\mathbf{u}, \mathbf{\Psi})^2 \, dt,
\] (2.9)
under the constraint \((\mathbf{\Psi}, \mathbf{\Psi}) = 1,\) leading to the eigenvalue problem
\[
\int \mathbf{R}_j(\mathbf{x}, \mathbf{x}') \mathbf{\Psi}_j(\mathbf{x}') \, d\mathbf{x}' = \lambda_j \mathbf{\Psi}_j(\mathbf{x}),
\] (2.10)
where \(\mathbf{R}_j(\mathbf{x}, \mathbf{x}') = \langle u_j(\mathbf{x}) u_j(\mathbf{x}') \rangle.\) The current field can then be expanded in terms of the eigenfunctions by
\[
u_j(\mathbf{x}, t) = \sum_k b_k(t) \mathbf{\Psi}_j(\mathbf{x}),
\] (2.11)
where \(b_k(t) = (\mathbf{u}, \mathbf{\Psi}_j).\) Note that both \(b_k\) and \(\mathbf{\Psi}\) are real.

As with the complex formulation, the eigenvalue problem (2.10) leads to a complete set of orthonormal eigenfunctions \((\mathbf{\Psi}_j),\) each corresponding to real positive eigenvalues \(\lambda_j\) that can be associated with the variance or energy in each mode. Following the notation defined earlier, (2.10) can be expressed in discrete form by
\[
\sum_{n=1}^N \mathbf{R}_j(\mathbf{x}_i, \mathbf{x}_m) \mathbf{\Psi}_j(\mathbf{x}_m) = \lambda_j \mathbf{\Psi}_j(\mathbf{x}_i),
\] (2.12)
where
\[
\mathbf{R}_j(\mathbf{x}_i, \mathbf{x}_m) = \frac{1}{N} \sum_{n=1}^N u_j(\mathbf{x}_i, t_n) u_j(\mathbf{x}_m, t_n).
\] (2.13)

Since \(\mathbf{R}_j(\mathbf{x}_i, \mathbf{x}_m)\) is a real symmetric matrix of size \(2M \times 2M,\) the solution of (2.12) yields \(2M\) real nonnegative eigenvalues and the same number of real eigenfunctions. In analogy with (2.7), the total energy in the flow must equal the sum of the eigenvalues: 
\[E = \text{Tr}(\mathbf{R}_j) = \sum_{m=1}^{2M} \lambda_m.\]

There are obvious mathematical differences between the complex and real-vector EOF analyses. First, the complex analysis reduces the vector nature of the data to that of a complex scalar, in contrast to the real analysis that leaves the vector structure intact. This approach naturally leads to different definitions for the variance given by (2.2) for the complex method and (2.9) for the real method. For example, it is evident that a true dot product is computed in (2.9), whereas in (2.2) the product of two complex scalars is calculated. Consequently, the expansions given by (2.4) and (2.11) for the complex and real analyses converge optimally, but only with respect to their corresponding variances defined by (2.2) and (2.9). It thus follows that the two methods need not yield the same results.

Specifically, Preisendorfer (1988) notes that the reduction to complex scalars employed in the complex EOF method incurs ambiguity in the direction of the resulting eigenstructures. The EOF method seeks to maximize the projection of the data onto the eigenstructures. If the vector nature of the flow field were maintained (i.e., if a real-vector analysis were employed) then this projection takes the form of a dot product between the data and the eigenstructure, thus fixing the direction of the eigenstructure. However, in the complex EOF analyses, the vector nature is lost, and the maximization of the data onto the eigenstructures takes the form of products between the complex data \(U\) and the complex eigenstructures \((\Phi_j).\) This allows for the calculation of eigenstructures that may be ambiguous in direction with respect to an external frame of reference and can thus be arbitrarily rotated (i.e., a constant phase shift applied). This was also noted by Kundu and Allen (1976); however, we mention that the phases (directions) of the eigenstructures are uniquely defined with respect to other modes. Other mathematical differences between the complex and real-vector analysis include the extent to which flow properties are retained by the eigenstructures, as explained in appendix A. Of particular importance, however, is the fact that there appears to be no way to extend the complex EOF method to three-dimensional velocity fields. In contrast, the real-vector analysis naturally extends to this multidimensional environment.

Solution of the problems defined by (2.3) and (2.10) requires finding the eigenvalues and functions of matrices whose size is of the order \(M \times M,\) where \(M\) is total number of spatial locations. It is not uncommon for \(M\) to be large; for example, the recent analysis of AVHRR sea surface data (Everson et al. 1997) dealt with datasets composed of \(M = 512^2 = 2.62 \times 10^5\)
Fig. 2. Several snapshots from the HIRES2 OSCR dataset (from Shay et al. 1995).

pixels, making it impossible to solve (2.3) or (2.10) directly. To avoid this difficulty, the problem can be formulated in such a way that the eigenfunctions can be written as a superposition of the temporal realizations or snapshots. This so-called method of snapshots (Sirovich 1987; see also appendix B) leads to eigenvalue problems in which the matrices are of the size of the number of temporal realizations \( N \). Thus, in cases where \( N \ll M \), the problem becomes computationally tractable. The method of snapshots yields a truncated set of eigenvalues and eigenfunctions that are completely equivalent to the direct solution [i.e., (2.3) and (2.10)]. For the dataset we will analyze, \( N \approx M \sim 10^4 \), which is small enough so that the eigenproblem can be solved directly. However, the method of snapshots has been chosen as the basis of our computer codes in anticipation of applications in which the spatial size of the datasets are much larger.

3. The experiment: Application to surface currents

a. OSCR dataset

The ocean surface current data were collected off the coast of North Carolina near Cape Hatteras during June and July 1993, as shown in Fig. 1. The study area is subject to many of the usual effects of tidal and wind forcing present over the shelf region but, in addition, is also subject to quasi-periodic incursions of high-speed, northeasterward flow from the Gulf Stream and southward flow of low-density water originating from Chesapeake Bay.
Fig. 3. (a) Eigenvalue spectrum for the complex EOF analysis. (b) Cumulative variance retention: complex EOF analysis.

The data were recorded using the ocean surface current radar (OSCR) system (Barrick et al. 1983). The OSCR system for this experiment consisted of two high-frequency radar transmit/receive stations operating at 25.4 MHz over a baseline of 24 km, representing the distance between the master and slave sites in Avon and Waves, North Carolina. At the utilized radar wavelength of about 11.8 m, backscattered energy from advancing and receding surface gravity waves had a wavelength of 5.9 m. Each site consisted of a 4-element transmit and 16-element, phased-array receiving array oriented at about 30° to the beachfront (Shay et al. 1995). This system forms an 85-m aperture that electronically steers a narrow beam over an illuminated area of about 30 km × 45 km, encompassing 700 grid points spaced 1.2 km apart. For each sample interval of about 20 min, two-dimensional surface current images were acquired over a 27-day period.

Figures 2a–d show some snapshots of the flow field. Overall, the flow showed a considerable amount of temporal and spatial variability. This variability included submesoscale to mesoscale eddies, convergence zones, and intrusions of the North Wall of the Gulf Stream over the shelf (Shay et al. 1995). In the western part of the domain, cooler coastal currents of 20 cm s⁻¹ were directed toward the south-southeast (Fig. 2a). Between the $B_1$ and $D_w$ moorings (denoted on the figures), surface currents converged where the offshore flow of 10–20 cm s⁻¹ at $D_e$ and $B_1$ was weak. On 23 June, a cyclonically rotating eddylike feature was located northeast of $B_1$ between the Gulf Stream and the coastal current (Fig. 2b). Maximum surface currents were 50–75 cm s⁻¹ at the confluence region of the coastal current and the Gulf Stream. The Gulf Stream moved farther offshore over the next 12 h. On 28 June, the Gulf Stream was located offshore as weak currents dominated the domain (Fig. 2c) forming a convergence zone and an eddy near $B_1$. Over the last two days of the experiment, the Gulf Stream penetrated farther onto the shelf with maximum speeds of 1 m s⁻¹, and a convergence zone was again located between $B_1$ and $D_w$ (Fig. 2d). The consistency between evolving snapshots indicated submesoscale surface current structure [scales less than the deformation radius of O(10 km)] due to the confluence of a coastal current with the Gulf Stream.

Shay et al. (1995) digitally filtered the data and performed harmonic analysis of tides in the time domain, revealing some details of the tidal and inertial flows. Shay et al. (1995) used bandpassing to isolate the different timescales in the data. Based on a spectral decomposition of the data, the 700 time series were separated into

$$U = \mathbf{u}_f + \mathbf{u}_t + \mathbf{u}_{in} + \mathbf{u}_{hf} + \mathbf{u}_r,$$

where the subscripts $lf$, $t$, $in$, $hf$, and $r$ refer to the low-frequency, tidal, inertial, high-frequency, and residual components, respectively. The low-frequency flow (timescale of 48 h or greater) accounted for 50%–70% of the total variance. As an alternative, the data were reexamined using the two EOF analysis methods described.

Both the real- and complex-valued EOF analyses were performed using the Naval Oceanographic Office Cray C90 supercomputer at Stennis Space Center, Mississippi. Both required linking with EISPACK routines for solving the resultant eigenvalue problem. The dataset was in no way modified prior to performing the EOF analysis. For example, we did not remove the mean current field and no filtering was performed. The real-valued analysis required 117 s of CPU time and about 32 Mw (megawords) of memory; the complex analyses, in contrast, required 486 s of CPU time and about 40 Mw of memory. As mentioned previously, we utilized the method of snapshots for both real and complex analyses, even though use of the direct methods of solution for both analyses would certainly have been tenable. Thus, the matrices for both the real and complex anal-
yses were 1902 × 1902 (i.e., the same as the number of temporal realizations, N).

b. Complex EOF results

The complex analysis yielded $M = 700$ unambiguous nontrivial eigenvalues, despite the fact that the matrix was 1902 × 1902, since the procedure will only yield as many nontrivial modes as the lesser of $M$ spatial points or $N$ temporal realizations. Thus, we obtain 700 unambiguous eigenvalues and 1202 trivial eigenvalues. This is evident in Fig. 3a, which shows an obvious drop-off in energy after mode 700.

The subsequent discussion pertains to the first five modes for both the complex and real-vector analysis. This is primarily because the first five modes retain 81%
of the total variance in the complex analysis and (as we will see in the next section) 77% of the variance in the real-vector analysis. Additionally, it is not clear how the characteristics of the small-scale structures evidenced in modes 6 and higher for either analysis are affected by resolution limitations of the OSCR system. In the higher modes, small-scale structures such as high-frequency internal waves (Shay 1996) may be present, as well as unresolvable turbulence and noise.

The convergence rate of the complex analysis (represented as a running total of percent variance retained with mode number) is shown in Fig. 3b. Twenty-four modes are required to reach the 90% level of total variance retained. For this complex EOF analysis we obtain orthonormalized \((\Phi, \Phi) = 1\) spatial eigenfunction maps. The time-dependent coefficients associated with each eigenfunction are conveniently split into their amplitudes and phases.
The first five modes of the complex analysis are shown in Figs. 4a–e; the maximum magnitude of any individual vector in each map is denoted in the caption to the figure. The spatial map for mode 1 (Fig. 4a) shows a nearly unidirectional shear flow, with the amplitudes of the vectors increasing in the offshore direction. This mode accounts for 66% of the total variance (Fig. 3b), which is similar to the variance in the low-frequency band in Shay et al. (1995); hence the amplitude plot closely resembles the low-frequency east and north currents found in that study (their Fig. 10). The amplitude series for this mode shows that mode 1 is relatively weak for the beginning of the experiment, and the phase plot indicates a negative phase (nearly $-\pi$ rad), which implies that the mode 1 flow was directed southwestward for about the first four days of the experiment. This is consistent with strong southward winds and the presence of cool coastal water during this period (see Fig. 4 in Shay et al. 1995) and with an offshore position for the Gulf Stream. For the remainder of the experiment, episodes of large-amplitude currents recur at 3–5-day intervals and the phase oscillates around a mean of 0.4 rad (22.91°). This indicates that the vectors in the modal map should be rotated about 23° in the positive (counterclockwise) direction from their present position to properly illustrate these strong flow events. The maps would then show northeastward flow increasing in speed toward the shelf break, as expected for Gulf Stream forcing. A rotary spectral analysis (not shown) of the mode 1 flow shows that the higher-frequency oscillations correspond to a clockwise-rotating (in time) semi-diurnal, M2 tidal current. The M2 tide (period of 12.42 h) appears in the mode 1 results because the complex analysis allows for rotation of the vectors and because the M2 currents must also be a fairly unidirectional flow with increasing speed offshore. We will return to this in section 3c.

Results for mode 2 are shown in Fig. 4b. The spatial map for this mode appears to show a strong diverging flow near the southeast section of the domain, but if the vectors are rotated by a mean angle of 23°, the map better corresponds to southwestward–northeastward shear flow. Strong southwestward inshore flow over the initial four days augments the mode 1 flow that corresponds to the influence of a large-scale coastal flow. The strong northeastward flow being found in only the offshore region of the domain points to a spatially limited Gulf Stream influence on the shelf at this time. A separate complex EOF analysis was done for the low-frequency data, which was generated after low-pass filtering the currents at 48 h by Eq. (3.1). The spatial map for mode 2 of this low-passed data resembled that of mode 2 of the present analysis. This low-passed mode 2 was found to have the highest correlation with the similarly low-passed wind (measured at the two discus buoys—DE and DW—shown in Fig. 1). This was done by first finding the projection of low-passed mode 2 at the OSCR cells closest to the buoys. The correlation with winds at the inshore buoy was 0.43, while it was only 0.40 for the wind measured farther offshore. This further supports identifying mode 2 with coastal, wind-driven flow at times when the Gulf Stream influence is relatively weak.

The remaining plots in Fig. 4 show the results for
modes 3–5. These maps show smaller-scale spatial structures and lower amplitudes. Taken together, they account for only 4% of the total variance in the data. Taken as they are (zero phase angle), the maps show limited areas of east–west-oriented areas of flow convergence that resemble convergence fronts found over the shelf by Graber et al. (1996) and Marmorino et al. (1997).

c. Real-vector EOF results

The real-vector EOF analysis yielded $2M = 1400$ [see Eqs. (2.12), (2.13)] unambiguous nontrivial modes and 502 modes that have effectively no energy. Again, this is due to the use of the method of snapshots in formulating temporal correlations over 1902 time steps. The truncation of the eigenvalue spectrum after mode 1400 is evident in Fig. 5a. Figure 5b shows a running total of variance retention as a function of mode number. The first mode retains about 61% of the total variance and the 90% convergence level is obtained with about 35 modes.

Figures 6a–e display the spatial structures and time-dependent coefficients for modes 1–5 for the real-vector EOF analysis. As with the complex EOF analysis, the spatial maps here are of orthonormalized eigenfunctions, that is, they satisfy $(\Psi, \Psi) = 1$. The time-dependent coefficients show the magnitudes and directions of the vectors; negative coefficients denote a 180° shift in direction relative to that shown on the spatial map. The spatial map for mode 1 shows a generally uniform shear flow, similar to that for complex mode 1 but with less curvature in the offshore direction. The temporal behavior for mode 1 (Fig. 6a) demonstrates a relatively strong southward flow near the beginning of the measurement period, which reverts to a succession of north–eastward flow events over the remainder of the time series, similar to the behavior of the first complex mode shown in Fig. 4a. Spectral analysis shows a broad peak centered at 3 days (similar to the complex case), as well as lesser peaks at 1.1 days and at the inertial and M2 frequencies. Thus, this shows that these higher-frequency components also have a southwest–northeast flow component.

The spatial structure of mode 2 (Fig. 6b) resembles the complex mode 2 result (after rotating it by a mean phase angle). It shows the strongest flow located at the offshore (southeast) edge of the domain, and some curvature of this flow onto the shelf to feed the southwestward flow on the inner shelf. The plot of amplitude shows almost all positive values so the map applies over most of the time series. Further, the mode 2 amplitudes tend to be largest when mode 1 amplitudes are least, again consistent with the influence of wind forcing as opposed to direct Gulf Stream effects.

Results for mode 3 (Fig. 6c) show a clear cross-shelf flow and highly periodic time behavior. Spectral analysis shows a dominant M2 peak and a weaker peak at the inertial frequency. The monthly cycle in tidal amplitude is also fairly clear, with stronger (spring) tidal flow at the beginning and end of the experiment and weaker (neap) flow near the middle (cf. Fig. 11 of Shay et al. 1995). Like complex mode 1, which we also identify with the M2 tide, the amplitudes of the current vectors increase in the offshore direction. The flow is obviously oscillatory along the northwest–southeast directions. This is consistent with the major axis of the M2 tidal ellipse being oriented in this onshore–offshore direction. Support for this is given by Berger et al. (1995), who show that currents at 5-m water depth in the area have ellipses oriented in the offshore direction and have major axis lengths of about 7–10 cm s$^{-1}$ (3.5–5 cm s$^{-1}$ in amplitude). A somewhat larger amplitude value of about 12.00 cm s$^{-1}$ can be derived from the mode 3 plot taking into account the “scale factor” (the maximum magnitude of the orthonormalized vectors in the spatial map; these are listed in the figure captions) and the root-mean-square value of the time-dependent coefficient in Fig. 6c.

The remaining modes (Figs. 6d,e) show smaller-scale
structures that together account for less than 2% of the total variance. Similar to the complex modes, they show regions of flow convergence and vorticity, but the real-vector maps seem to show more of a cellular structure of closed eddies (see also below). These eddies tend to be stretched or elongated in the alongshore (southwest–northeast) direction, especially in the offshore part of the domain. The amplitude behavior shows a mix of low and high frequencies. This is especially evident for mode 4, which has spectral peaks centered at about four days and near the inertial frequency. In particular, note the clear near-inertial oscillations occurring in Fig. 6d over days 7–9, in agreement with Shay et al. (1995) and Shay 1996.
d. Comparisons

Several inherent differences (discussed previously and in appendix A) between the two EOF analysis methods exist that preclude expectations of identical results. However, some parallels between the two methods can be drawn based on the results. For example, the real-vector analysis method was able to resolve two strongly unidirectional modes with orthogonal spatial structures (modes 1 and 3), while the complex EOF analysis did not perform this decoupling. One likely reason for this can be found in the similarities in spatial structure between mode 1 of the complex EOF analysis and modes 1 and 3 of the real-vector analysis. All three have a strong unidirectionality (the only unidirectional modes in either analysis). Note that mode 1 of the complex analysis includes the tidelike signal shown in mode 3 of the real analysis, so that the real-vector analysis seems to decouple flows that are spatially orthogonal to...
each other (modes 1 and 3 of the real-vector analysis). This decoupling is not evident in the results of the complex EOF analysis. Indeed, orthogonal unidirectional flows can be represented in the complex EOF analysis as rotations of the vectors (changes in the phase).

Another distinguishing characteristic between the complex and real-vector analyses is the curvature found in the spatial maps. As noted before, both modes 1 and 3 of the real-vector analysis are fairly unidirectional, with no strong curvature characteristics. Additionally, eddylike structures in the higher modes of the real analysis appear as complete loops; this can be seen in mode 5 (Fig. 6e), for example. In contrast, the spatial maps of the modes from the complex analysis do not show any unidirectional trends (save for mode 1) and the eddylike structures that appear do not appear closed (e.g., closed streamlines). Each mode must have some degree of curvature, and thus it is possible that slablike flows and eddylike flows are both represented in each mode (at least the lower five modes), with the unidirectional shear flow smearing the eddy features. This may be an indication that, for very complicated flows, the complex analysis overconstrains the analysis and does not admit modes that appear to correspond well to one’s intuitive picture of the flow. The fact that the real-vector analysis clearly separates the tidal motion hidden in the complex analysis seems to indicate this. Another reason (see appendix A) is that the real analysis preserves incompressibility, while the complex method does not. Since the present (2D) flow is incompressible (with the exception of localized convergent or divergent regions), we should expect each individual real-vector eigenmode to correspond more closely to a physically realizable flow, which appears to be the case here.

The interpretation of the eddylike structures in the higher modes is not entirely clear. There are several possible explanations. One is that preexisting eddies on the shelf are being strained by the lower-frequency flow (as represented by the lowest mode). Another is that horizontal velocity fine structure preexisting along the edge of the Gulf Stream is advected through the study area as the stream moves on and offshore. A third explanation may be that the cellular structure (such as in the real-vector modes 4 and 5) combines to allow traveling waves. This is possible if the map structures are a quarter-wavelength apart and the amplitude series a quarter-cycle out of phase (e.g., Everson et al. 1996); modes 4 and 5 exhibit some of this behavior. An animation of the vorticity contained in the EOF modes (Kaifatu et al. 1996) was made to help address these ideas. It shows a rather confusing mixture of eddies and linear structures moving onshore and alongshore through the domain. However, there were no clearly stationary eddies such as might be trapped by local shelf bathymetry. Clearly, more work needs to be done to sort out these possible mechanisms that may be inextricably linked in these EOF analyses because of the similar spatial scales of the dominant physical processes.

4. Conclusions
We have compared two EOF analysis methods in this study—the complex EOF method and the real-vector EOF method—by using them both on the same ocean
current field dataset. The complex method reduces the components of the vector field to a complex scalar, whereas the real-vector EOF method takes into account the vector nature of the current field data in a direct manner. Several mathematical differences were noted between the complex and real-vector EOF methods, particularly with respect to the definitions of variance in each method. In the real-vector method, since the vector nature of the flow field is retained, the variance is defined with respect to a dot product [Eq. (2.9)], whereas in the complex EOF method the product of two complex numbers is computed [Eq. (2.2)]. Therefore, each method is optimal but only with respect to the corresponding definition of energy or variance precluding any expectations of obtaining identical results. Additionally, because of the loss of the vector nature of the flow field with the complex analysis, there may be some directional ambiguity in the complex EOF results.

We applied the techniques to the ocean surface current dataset of Shay et al. (1995). The primary objective was not to extract all the physics from the modes but to note the differences in the results and what the differences could mean physically. In this particular case, it seemed that the real-vector analysis was better able to separate flows of differing temporal/spatial scales that were orthogonal to each other (for instance, the alongshelf Gulf Stream shear and the cross-shelf tidal flow) than the complex EOF analysis. Additionally, the complex EOF analysis results contained only one mode with a unidirectional flow; the remaining modes had some curvature, with the eddylike structures in these modes consisting of open loops. In contrast, the real-vector analysis contained two modes with domainwide unidirectionality, but they were orthogonal in direction to each other. The mode with cross-shelf orientation in this real-vector analysis had a time-dependent coefficient that oscillated at a frequency very close to the diurnal tidal cycle. Higher modes in the real-vector analysis had eddylike structures that consisted of closed loops, leading to the speculation that this analysis may be better able to separate unidirectional shelfwide flow from curvilinear eddylike flow. The complex analysis, on the other hand, seemed to have both unidirectional and eddylike flows in each of the modes displayed here with the background shear smearing the eddy structures. The main intention was to reintroduce the real-vector EOF analysis to the application to oceanographic processes and to investigate how the results from this technique differs from those from more established techniques. However, it is not unreasonable to hypothesize that many of the distinguishing features noted here between the complex and real-vector EOF analyses would likely also be present in similar analyses of other surface current datasets such as Duck94 (Shay et al. 1998) and the Florida Keys (Shay et al. 1997).

Acknowledgments. Most of the research was performed when JMK was a JOI/CORE Postdoctoral Fellow in the Remote Sensing Division at the Naval Research Laboratory in Washington, D.C.; follow-up work done by JMK was funded by the Office of Naval Research (ONR) through the 6.2 Coastal Simulation NRL Core project. RAH and GOM were supported by the High Resolution Remote Sensing ARI, funded by ONR. LKS was supported by ONR and the Minerals Management Service for the acquisition and processing of the surface current measurements through N00014-91-J-4133 and the analysis of the data through N00014-96-1-0111 by the ONR Remote Sensing Program. The computational work was performed on resources made available by the Department of Defense High Performance Computing Center. The authors wish to acknowledge many useful discussions with Prof. Larry Sirovich (Brown University) regarding various aspects of this work, particularly regarding his contributions to appendix A.

APPENDIX A

Mathematical Properties of Complex and Real-Vector EOF Analyses

It is important to understand the extent to which a complex EOF analysis preserves the properties of the original current field. In this context, it remains unclear whether the eigenfunctions preserve incompressibility or irrotationality. Suppose, for example, that the current field is incompressible and satisfies \( \nabla \cdot \mathbf{u} = 0 \). For the real-vector analysis, taking the divergence of (2.10) gives

\[
\int \langle \partial_t \mathbf{u} \rangle \Phi_j = \lambda \partial_t \Phi_j. \tag{A1}
\]

Since \( \mathbf{u} \) is divergenceless, \( \langle \partial_t \mathbf{u} \rangle = 0 \), so \( \partial_t \Phi_j = 0 \). Thus, the real-vector eigenfunctions preserve incompressibility. It is also clear that the eigenfunctions will preserve irrotationality, that is, \( \nabla \times \mathbf{u} = 0 \) implies \( \nabla \times \Phi = 0 \).

A similar analysis can be performed for the complex analysis. First, (2.3) is decomposed into real and imaginary parts:

\[
\int \langle uu' \rangle \Phi'_{u'} + \int \langle uv' \rangle \Phi'_{v'} + \int \langle vv' \rangle \Phi'_{v'} - \int \langle vu' \rangle \Phi'_{u'} = \lambda \Phi_u \tag{A2a}
\]

and

\[
\int \langle uu' \rangle \Phi'_{u'} + \int \langle uv' \rangle \Phi'_{v'} - \int \langle vv' \rangle \Phi'_{v'} + \int \langle uu' \rangle \Phi'_{v'} = \lambda \Phi_v. \tag{A2b}
\]

where the superscript prime in (A2b) indicates variables...
over which the integration is taken. Taking \( \frac{\partial}{\partial x} \) of (A2a) and adding this to \( \frac{\partial}{\partial y} \) of (A2b) yields

\[
\lambda (\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}) = - \int \langle (u_x + v_y) \rangle \Phi_x + \int \langle (u_x - v_y) \rangle \Phi_y
\]

\[
+ \int \langle (u_x + u_v) \rangle \Phi_x + \int \langle (u_x - u_v) \rangle \Phi_y. \quad (A3)
\]

If \( \nabla \cdot \mathbf{u} = 0 \), the first two terms in (A3) will vanish, leaving the terms involving \( u_x = \Omega, \) where \( \Omega \) is the vorticity. Thus, the complex eigenfunctions will be incompressible (they satisfy \( \partial \Phi_x / \partial x + \partial \Phi_y / \partial y = 0 \)), if the current field is both incompressible and irrotational. Furthermore, it can be shown that if \( \nabla \cdot \mathbf{u} = 0 \) and \( \nabla \times \mathbf{u} = 0 \), then the complex eigenmodes will also be incompressible and irrotational. Unlike the real-vector analysis, a divergenceless current field will not alone ensure divergenceless eigenmodes, which is the significant finding here. In summary, the complex eigenmodes will be divergence free if the current field is both divergenceless and irrotational, in which case the eigenmodes will also be irrotational.

For purposes of illustration, it is useful to consider the differences between these two analyses for flows confined to a rectangular domain in some elementary cases. First, we consider a unidirectional flow, \( u = u(x, y), \) and \( v = 0 \). It is evident that the eigenfunctions for the real-vector analysis in this case are given by \( \Psi = (\Psi_x, 0) \), that is, they will also be unidirectional. For this case, the eigenvalue problem for complex analysis (A2) reduces to

\[
\int \langle uu' \rangle \Phi_x = \lambda \Phi_x, \quad (A4a)
\]

and

\[
\int \langle uu' \rangle \Phi_y = \lambda \Phi_y. \quad (A4b)
\]

It follows that \( \Phi_x = \Phi_y \) is a possible solution. Furthermore, since (A4a) also applies to the real-vector analysis, it follows that \( (\Phi_x, \Phi_y) = (\Psi_x, \Psi_y) \). Therefore, for a unidirectional flow, the complex eigenfunctions do not preserve unidirectionality in contrast to the real-vector eigenfunctions. A second case is that of a time-independent flow field given by \( u = u(x, y) \) and \( v = v(x, y) \), a flow consisting of one snapshot. In this case the eigenfunctions for the real-vector case are given by \( \Psi_u = au \) and \( \Psi_v = av \), and for the complex case by \( \Phi_u = bu + cv \) and \( \Phi_v = bv - cu \) where \( a, b, \) and \( c \) are arbitrary real constants. Thus, the complex modes are linear combinations of each component of the current, whereas for the real case, the eigenmodes point strictly in the direction of the current.

An important issue is whether there exists certain classes of flows for which the complex analysis is equivalent to the real analysis. To approach this issue, equivalence exists if the last two terms in (A2a) and (A2b) are zero. If the current field is both incompressible and irrotational, then these terms reduce to \( b_x = \langle u \rangle \) and \( b_y = \langle v \rangle \), where

\[
B = \int_0^L [A(L, y) \Phi_x(L, y) - A(0, y) \Phi_x(0, y)] dy
\]

\[
+ \int_0^L [A(x, L) \Phi_x(x, L) - A(x, 0) \Phi_x(x, 0)] dx, \quad (A5)
\]

and where \( A \) is the streamfunction for the velocity field. These terms contain boundary values that result from an integration by parts over the domain. It is evident that if \( \mathbf{u} = 0 \) on the boundary, then also \( \Phi_x = \Phi_y = 0 \) on the boundary and therefore \( b_1 = b_2 = 0 \). If this is the case, however, \( u = 0 \) everywhere for the potential flow being considered here. This leads to the conclusion, since we cannot ensure the vanishing of the terms given in (A5) for any nonzero flow, that the complex and real-vector eigenfunctions are not equivalent even for the case of potential flow. As a final remark we reiterate that there appears to be no way to extend the complex method to three dimensions, which is in striking contrast to the real analysis.

**APPENDIX B**

**Method of Snapshots**

The method of snapshots referred to in section 2 is briefly described. First, we observe that (2.12) and (2.13) can be rewritten, by changing the order of summation, as

\[
\frac{1}{N} \sum_{n=1}^{N} u_j(x_n, t_j) a^{(i)}_n = \lambda^{(i)} \Psi^{(i)}_j(x_j), \quad (B1)
\]

where

\[
a^{(i)}_n = \sum_{m=1}^{M} u_j(x_n, t_j) \Psi^{(i)}_j(x_m). \quad (B2)
\]

Using (B1) to derive an expression for \( \Psi^{(i)}_j(x_m) \) and substituting this back into (B2) gives

\[
\sum_{n=1}^{N} C_{in} a^{(i)}_n = \lambda^{(i)} a^{(i)}_n, \quad (B3)
\]

where

\[
C_{in} = \frac{1}{N} \sum_{n=1}^{N} u_j(x_n, t_j) u_j(x_m, t_j). \quad (B4)
\]

Solving (B3) for the eigenvalues \( \lambda^{(i)} \) and coefficients \( a^{(i)}_n \), the \( \Psi^{(i)}_j \) can be recovered by using (B1), which states that the spatial eigenfunctions are weighted sums of the original set of realizations or snapshots, \( u_j(x_n, t_j) \). It
may be that the number of spatial locations \( M \) is so large that numerical solution of (2.12) is impossible. It can be seen from (B4), however, that when \( N \ll M \) (i.e., when the number of snapshots is significantly less than the number of spatial nodes), it is possible to solve the problem exactly for the first \( N \) modes. Thus, the method of snapshots allows for a partial solution to very large problems within the bounds of current computational resources.

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