Do Gravity Waves Transport Angular Momentum Away From Tropical Cyclones?

Yumin Moon* and David S. Nolan

Rosenstiel School of Marine and Atmospheric Science
University of Miami
Miami, FL

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*Corresponding author address:
Yumin Moon, University of Miami, RSMAS/MPO
4600 Rickenbacker Causeway, Miami, FL 33149, USA.
e-mail: ymoon@rsmas.miami.edu
Abstract

Previous studies have suggested that gravity waves can transport a significantly large amount of angular momentum away from tropical cyclones, as much as 10% of the core angular momentum per hour. These previous studies used the shallow water equations to model gravity waves radiating outward from rapidly rotating inner core asymmetries. This issue is re-investigated with a three-dimensional, nonhydrostatic, linear model of the vortex-anelastic equations. The response of balanced, axisymmetric vortices modeled after tropical cyclones to rotating asymmetric heat sources is examined to assess angular momentum transport by gravity waves radiating away from the core region of the vortices. Calculations show that gravity waves do transport angular momentum away from the vortex core, but an amount that is several orders of magnitude smaller than recent estimates.
1. Introduction

Convection and subsequent latent heat release is the dominant energy source for tropical cyclones; therefore it is critical to understand their full effect on tropical cyclones. Whenever there is convective activity and latent heat is released, rapid adjustment processes disperse the source of imbalance and bring the atmosphere back into a balanced state. In previous studies that examined the effect of convection on tropical cyclones, the role of gravity waves generated from these adjustment processes was neglected because it was assumed that the end product of the rapid adjustment processes due to the introduction of unbalanced thermal perturbations could be represented solely as vorticity perturbations. For this reason, these studies (e.g., Carr and Williams 1989; Montgomery and Kallenbach 1997; Nolan and Farrell 1999) chose balanced and two-dimensional frameworks to simulate the response of a balanced vortex to asymmetric convection. In addition, vortex Rossby waves are dynamically more active in terms of wave activities than gravity waves, as shown in the empirical normal modes analysis of a numerically simulated tropical cyclone by Chen et al. (2003).

However, recent studies by Nolan and Grasso (2003) and Nolan et al. (2007) have shown that capturing the detailed evolution of rapid adjustment processes such as gravity wave radiation is qualitatively and quantitatively important in understanding the full effect of convection on tropical cyclones. Gravity waves are ubiquitous in tropical cyclones as they are continuously generated from convection and the ensuing adjustment processes. Even when active convection is absent, rotating asymmetries in the eyewall can create rapid potential vorticity (PV) fluctuations, generating gravity waves. Chimonas and Hauser (1997) examined gravity waves in the context of a mesocyclone in a mean static atmosphere and estimated that gravity waves could be very effective in removing angular momentum from the mesocyclone. Chow et al. (2002,
hereafter CCL02) argued that large-scale moving outer spiral bands, often seen in full-physics numerical simulations, can be modeled by waves emanating from a rotating elliptical vortex, similar to the Lighthill radiation mechanism (Lighthill 1952; Ford 1994). Chow and Chan (2003, hereafter CC03) derived an analytical expression based on the shallow water equations of CCL02 to show that gravity waves can transport a significant amount (~ 10% per hour) of angular momentum away from the tropical-cyclone-like vortex.

Given the significantly large magnitude of the angular momentum transport estimated by CC03, it seems worthwhile to re-examine how efficient gravity waves might be in transporting angular momentum away from tropical cyclones. First, we provide a brief review of CCL02 and CC03 in section 2. Section 3 introduces the numerical model and the basic state vortices to be used with it. Section 4 describes the structure of rotating heat sources, examines the evolution of a control case simulation, and presents a method to calculate the amount of angular momentum transport by gravity waves. Section 5 assesses the sensitivity of the control case to different parameters. Discussion and summary are provided in sections 6 and 7 respectively.

2. A review of CCL02 and CC03

CCL02 first performed an idealized numerical simulation of a tropical cyclone with the fifth-generation Pennsylvania State University/National Center for Atmospheric Research Mesoscale Model (PSU/NCAR MM5) by initializing the wind field with an axisymmetric vortex in a quiescent environment. During the simulation, an elliptical core and large-scale moving spiral bands developed (Fig. 1a). Such features have been observed and documented in real tropical cyclones (e.g., Hurricane David (1979) in Willoughby et al. 1984; and Typhoon Herb (1996) in Kuo et al. 1999, see Fig. 1b). CCL02 then argued that those similar spiral band features
(Fig. 1c) can be generated from a rotating elliptical vortex in the shallow water equations through the Lighthill radiation mechanism, assuming that the basic state of the vortex in addition to its perturbations has a small Froude number, which is defined as

\[ Fr = \frac{v}{\sqrt{gh_0}}. \]  

(1)

In Eq. (1), \( g \) is the gravitational acceleration, \( h_0 \) is the background equivalent depth, and \( v \) is the velocity scale of the basic state vortex. The Froude number is less than unity so the flow is subcritical. Based on the shallow water equations of CCL02, CC03 derived an expression for the radial current of angular momentum carried away by shallow water gravity waves and showed by using an Eliassen-Palm theorem that this radial current of angular momentum for quasi-steady shallow water gravity waves remains unchanged during their propagation through the axisymmetric mean flow. After some simplifications, CC03 arrived at an expression for the loss rate of angular momentum (\( \dot{L} \)) of the tropical cyclone vortex:

\[
-\frac{\Delta L}{L} \geq \frac{m^2 \pi^2 \varepsilon^2 A^2}{16} \left( \frac{\Omega_p a c}{c} \right) \left( \frac{\zeta_0}{\Omega_p} \right). \]  

(2)

In Eq. (2), \( m \) is the azimuthal wavenumber of the waves; \( \Omega_p \) is the angular velocity of the waves; \( \zeta_0 \) is the central vorticity of the vortex; \( a \) is radius; \( \varepsilon \) is the ratio of semi-minor axis to semi-major axis; \( c \) is the radial propagation speed of the spiral gravity waves; and \( A \) is a measure of vorticity distribution inside the elliptical vortex. CC03 evaluated Eq. (2) with the depth-averaged MM5 model output and estimated that the angular momentum transport by gravity waves can remove 13% of the angular momentum of the storm core per each revolution (~ 75 minutes) of the spiral band features. This is a very significantly high loss rate. Since three-dimensional adjustment processes were neglected in CCL02 and CC03, this issue of radiating gravity waves and their
angular momentum transport in tropical cyclones is examined here in this study with a three-dimensional, nonhydrostatic, and linear model.

3. The model and basic state vortices

a. The numerical model: 3DVPAS

To investigate this problem, we use a three-dimensional, nonhydrostatic, linear numerical model of vortex dynamics, now known as Three Dimensional Vortex Perturbation Analysis and Simulation (3DVPAS). This model allows for the simulation and analysis of perturbations (either purely asymmetric or symmetric) on axisymmetric, balanced vortices, and it is based on the vortex-anelastic (VA) equations on cylindrical $f$-plane coordinates, as derived in Hodyss and Nolan (2007). Perturbations have arbitrary structures in the radial and vertical directions but vary harmonically in the azimuthal direction. The 3DVPAS model is linear and nonhydrostatic, and it can simulate both asymmetric and symmetric motions, with some coupling between them by using the eddy flux divergences arising from asymmetric motions as forcing for symmetric motions. Free-slip, solid-wall boundary conditions are enforced on all sides, and Rayleigh damping regions (sponges) are placed at the upper and outer boundaries to suppress the reflection of gravity waves. The domain size for all simulations in this study is 300 km in the radial direction and 20 km in the vertical direction. The grids are regularly spaced in the vertical direction with the grid size of 0.5 km but stretched in the radial direction so that the grid size is about 2 km in the inner core region ($0 < r < 80$ km) but larger (~6 km) in the far outer core region with a smooth transition in between. Further details of 3DVPAS can be found in Nolan and Montgomery (2002), Nolan and Grasso (2003), and Nolan et al. (2007).
b. Basic state vortices

Despite the importance of frictionally induced secondary circulations, tropical cyclones can be approximated as vortices in hydrostatic and gradient wind balance with their respective pressure and temperature fields (Willoughby 1990). Thus, secondary circulations will be neglected in all of the basic state vortices considered in this study, but their tangential wind fields are constructed realistically, albeit idealized.

A number of profiles have been used in the past to represent the radial structure of the tropical cyclone tangential wind field. One popular choice is a Gaussian (GS) vorticity profile,

$$\zeta(r) = A \times \exp\left\{-\left(\frac{r}{b}\right)^2\right\}, \quad (3)$$

where $A$ is the maximum amplitude of vorticity, and $b$ is a parameter that controls the width of the GS distribution. However, as shown by Mallen et al. (2005), idealized vortices constructed with the GS profile have an unrealistic representation of the tangential wind field outside the radius of maximum wind (RMW). The GS profile outside the RMW decays as $1/r$. However, the observed wind fields show a slower decay rate, often as $1/r^a$, where $1/3 < a < 2/3$ (e.g., Shea and Gray 1973; Mallen et al. 2005).

There are alternatives to remedy this problem. One possibility is the “smoothed” Rankine-with-skirt (RWS) profile, in which the high vorticity core is surrounded by a region of lower vorticity, as used Nolan et al. (2007). Its vorticity field is defined by

$$\tilde{\zeta}(r) = \begin{cases} 
\zeta_1, & 0 \leq r \leq r_1 - d_1 \\
\zeta_2 + \left(\zeta_1 - \zeta_2\right)S[(r - r_1 + d_1)/2d_1], & r_1 - d_1 \leq r \leq r_1 + d_1 \\
\zeta_2, & r_1 + d_1 \leq r \leq r_2 - d_2 \\
\zeta_2 S[(r - r_2 + d_2)/2d_2], & r_2 - d_2 \leq r \leq r_2 + d_2 \\
0, & r_2 + d_2 \leq r 
\end{cases}, \quad (4)$$
where \( S(x) = 1 - 3x^2 + 2x^3 \) is the cubic Hermite polynomial which satisfies \( S(0) = 1, S(1) = 0, \) and \( S'(0) = S'(1) = 0. \) \( \zeta_1 \) and \( \zeta_2 \) represent the values of high vorticity in the core and lower vorticity surrounding the core, and \( r_1 \) and \( r_2 \) are the centers of the transition regions whose half-widths are \( d_1 \) and \( d_2. \)

The Modified Rankine (MR) profile is another good alternative and has a tangential velocity field defined by

\[
\tilde{v}(r) = \begin{cases} 
v_{\text{max}} \left( \frac{r}{R_{\text{MW}}} \right)^a, & r \leq R_{\text{MW}} \\
v_{\text{max}} \left( \frac{R_{\text{MW}}}{r} \right)^a, & r > R_{\text{MW}} 
\end{cases}
\]

(5)

where \( v_{\text{max}} \) is the prescribed maximum tangential velocity and \( a \) is the decay parameter. Another possibility is the single-exponential sectionally continuous (SC) profile from Willoughby et al. (2006), which is a piecewise continuous wind profile, modeled after tropical cyclone wind field measurements by the National Oceanic and Atmospheric Administration and U.S. Air Force aircraft. The SC profile has the wind increasing with a power of radius inside the eye then decaying exponentially outside the eye. The SC tangential wind field is defined as

\[
\tilde{v}(r) = \begin{cases} 
v_i = v_{\text{max}} \left( \frac{r}{R_{\text{max}}} \right)^n, & (0 \leq r \leq R_i) \\
v_i(1-w) + v_0 w_i (R_i \leq r \leq R_2) \\
v_0 = v_{\text{max}} \exp \left( -\frac{r - R_{\text{max}}}{X_1} \right), & (R_2 \leq r)
\end{cases}
\]

(6)

where \( w \) is a weighting polynomial

\[
w(\xi) = 126\xi^5 - 420\xi^6 + 540\xi^7 - 315\xi^8 + 70\xi^9,
\]

(7)
in terms of a nondimensional parameter $\xi$, where $\xi = (r-R_1)/(R_2-R_1)$; if $\xi \leq 0$, $w = 0$ and if $\xi \geq 1$, $w = 1$. The values of $n$, $R_{\text{max}}$ and $X_i$ in Eq. (6) are determined empirically by the prescribed values of $v_{\text{max}}$, $\varphi$, and $R_2-R_1$.

Unlike the radial structure of the tropical cyclone wind field, its vertical structure is not nearly as well documented. However, a somewhat realistic tropical cyclone wind field can be obtained by extending the surface tangential wind profile into the vertical direction by using a function such as

$$v(r,z) = \tilde{v}(r) \exp \left( - \frac{|z-z_0|^\alpha}{\alpha L_z} \right),$$

where $z_0$ is the altitude of the maximum wind speed; $L_z$ is the factor that controls the depth of the barotropic part of the vortex; and $\alpha$ is the decay rate above the barotropic zone. The values of $z_0$, $L_z$, and $\alpha$ are chosen through trial and error to construct realistic-looking tropical cyclone wind fields; $z_0 = 0$ m, $L_z = 6250$ m, and $\alpha = 1.9$ for all vortices in this study. After constructing the tangential wind field of the basic state vortices, the pressure and temperature fields that hold the vortex in hydrostatic and gradient wind balance in the Jordan mean sounding are computed. This is achieved by an iterative procedure described in Nolan et al. (2001) using the Jordan (1958) mean hurricane season sounding as the far field environment. Fig. 2 shows the properties of four different basic state vortices (MR, RWS, GS, and SC) which are created by using the parameters listed in Table 1. All vortices have the maximum tangential velocity of $40 \text{ m s}^{-1}$ at the RMW of 29 km, and they are statically and symmetrically stable.

4. The evolution of a control case simulation - MR vortex

a. Structure of rotating convective heat sources
Radar observations of elliptical eyes in tropical cyclones (e.g. Kuo et al. 1999; see their Fig. 1b) usually show reflectivity maxima at the ends of the ellipse, suggesting that the ellipse is maintained in part by asymmetric convective heating. Examples of higher azimuthal wavenumber \((n > 2)\) asymmetries can also be found, as in Fig. 1 of Lewis and Hawkins (1982) and Fig. 4 of Muramatsu (1986). Therefore, we propose here that a tropical cyclone with an elliptical eye may be approximated as the superposition of rotating inner core convective asymmetric heat sources on a balanced axisymmetric basic state vortex. In 3DVPAS, this can be modeled by imposing purely asymmetric azimuthal wavenumber-two heating perturbations on the eyewall region of a basic state vortex.

In this study, gravity waves are generated by treating convective asymmetries as forcing of the damped oscillator. However, gravity waves are also likely to be generated by the natural modes inherent to the structure of the vortex. Convective heating controlled by such modes can either encourage or inhibit gravity wave emissions (Schecter and Montgomery 2004, 2006, 2007). Here, the objective of this study is to evaluate the estimate by CCL02 and CC03 which examined quasi-steady radiating gravity waves. A straightforward way to generate a field of steadily radiating gravity waves is to treat convective asymmetries as independent forcing.

The diabatic heat sources have a “Gaussian-like” structure in the radial and vertical directions,

\[
\dot{Q}_n(r, \lambda, z, t) = \dot{Q}_{\text{max}} \exp \left\{- \left( \frac{r - r_b}{\sigma_r} \right)^2 - \left( \frac{z - z_b}{\sigma_z} \right)^4 \right\} \exp(\mathbf{i}n\lambda - \mathbf{i}n\omega t), \tag{9}
\]

where \(r_b\) and \(z_b\) are the radial and vertical center locations, and \(\sigma_r\) and \(\sigma_z\) are the radial and vertical half-widths of the heat sources. The fourth power exponential variation is used in the vertical to keep the heating more sharply confined vertically. Unless noted otherwise, all
calculations in this study use \( r_b = \text{RMW} = 29 \text{ km}, \ z_b = 5 \text{ km}, \ \sigma_r = 3 \text{ km}, \) and \( \sigma_z = 2 \text{ km}. \) The complex exponential term in Eq. (9) rotates the heat sources around the center of the storm at an angular velocity, \( \omega = \frac{v_{\text{rot}}}{r_b} \), where \( v_{\text{rot}} \) is the rotation speed. Due to the large vorticity gradient just outside the eye, the rotating asymmetries of the eye can be characterized as the propagation of edge vortex Rossby waves (Kuo et al. 1999; Reasor et al. 2000). From the linear wave theory on a Rankine vortex (Lamb 1932), the rotation speed of edge vortex Rossby waves is given by

\[
v_e = v_i \left( 1 - \frac{1}{n} \right), \quad (10)
\]

where \( v_i \) is tangential velocity at \( r = \text{RMW} \) and \( n \) is azimuthal wavenumber.

The diabatic heat sources are purely asymmetric with zero net heating. By combining the entropy equation,

\[
\dot{\theta} = \frac{\theta}{\rho c_p T} \dot{Q}, \quad (11)
\]

and the definition of potential temperature,

\[
\theta = T \left( \frac{p_0}{p} \right)^{\gamma/c_p}, \quad (12)
\]

the diabatic heating rate (\( \dot{Q} \)) can be converted to potential temperature tendency (\( \dot{\theta} \)) by using

\[
\dot{\theta} = \frac{1}{\rho c_p} \dot{Q} \left( \frac{p_0}{p} \right)^{\gamma/c_p}. \quad (13)
\]

The value of \( \dot{Q} \) is prescribed to result in the desired maximum \( \dot{\theta} \).

In numerically simulated tropical cyclones (e.g., Zhang et al. 2002; Hendricks et al. 2004; Montgomery et al. 2006), the diabatic heating field is often found to be highly asymmetric. Typical maximum values of azimuthally averaged heating rates range between 50 K hr\(^{-1}\) and 80 K hr\(^{-1}\) for numerically simulated tropical cyclones with the maximum tangential wind speeds
greater than 50 ms\(^{-1}\). Here we wish to represent the heating associated with long-lasting inner core convective asymmetries (e.g., Kuo et al. 1999; Reasor et al. 2000). Assuming that a reasonable estimate of the azimuthally averaged heating rate for a tropical cyclone with \(v_{\text{max}} = 40\) ms\(^{-1}\) ranges between 20 K hr\(^{-1}\) and 40 K hr\(^{-1}\) and that the appropriate magnitude of rotating convective asymmetries is 25% of the axisymmetric mean, then the maximum value of \(\dot{\theta}\) is between 5 K hr\(^{-1}\) and 10 K hr\(^{-1}\). In addition, numerically simulated tropical cyclones show that vertical velocities associated with gravity waves are on the order of 0.1 ms\(^{-1}\). To generate gravity waves of this magnitude, it is required to impose the heat sources of 5 K hr\(^{-1}\) to 10 K hr\(^{-1}\) heating on the basic state vortices of \(v_{\text{max}} = 40\) ms\(^{-1}\) in 3DVPAS. Unless noted otherwise, the maximum \(\dot{\theta}\) is chosen to be 5 K hr\(^{-1}\) for all simulations in this study.

\[ \text{b. Evolution of rotating convective asymmetries on the MR vortex} \]

We now present the evolution of the rotating asymmetries on the MR vortex as the control case. The \(n = 2\) convective heat sources are purely asymmetric and centered at \(r = \text{RMW}\) (see Fig. 3). In accordance with Eq. (10), they are set to rotate at 50% of the local tangential mean flow at \(r = \text{RMW}\), so \(v_{\text{rot}} = 20\) ms\(^{-1}\).

First, we examine how these rotating asymmetric convective heat sources evolve over time. Fig. 4 shows the time evolution of kinetic energy (KE) and available potential energy (APE) of the asymmetric response during the simulation. Expressions for KE and APE of baroclinic, asymmetric \((n > 0)\) perturbations on baroclinic, axisymmetric vortices are

\[
KE_n = \int \frac{\rho}{2} (\bar{u}_n^2 + \bar{v}_n^2 + \bar{w}_n^2) r^2 dr dz, \quad (14)
\]

\[
APE_n = \int \frac{1}{2} \frac{\rho}{\bar{N}^2} \bar{\theta}_n^2 r^2 dr dz, \quad (15)
\]
as derived in sec. 2b of Nolan et al. (2007). Variables with subscript are the complex functions in the $r$-$z$ plane that contains the amplitude and phase of the perturbations. Overbars over the product of subscripted variables refer to the averages around the azimuth of the real parts of the complex functions, therefore $\overline{a_n b_n} = 0.25 \times (a_n^* b_n + a_n b_n^*)$. $R$ and $Z$ in Eqs. (14) and (15) denote the radial and upper boundaries of the domain. It seems clear that a nearly steady state is reached after $t = 10$ h. In a first examination of the vertical velocity field at $t = 24$ h (Figs. 5a and b), radiating gravity waves are not clearly evident because the asymmetric response is strongly localized near the eyewall region. However, focusing on the outer core region (Figs. 5c and d) by suppressing the data between $r = 0$ km and $r = 2 \times RMW$ clearly shows that gravity waves are generated and propagate away from the vortex core in a spiral fashion.

Since unbalanced heating perturbations are placed near the eyewall region in the presence of a vorticity gradient, both gravity waves and vortex Rossby waves are excited. One way to discern vortex Rossby waves from gravity waves is to examine the PV field of the asymmetric response. Since the magnitude of the vortical (PV) component of wave activity associated with vortex Rossby waves is significantly larger than that of gravity waves (Chen et al. 2003), vortex Rossby waves would appear more prominently on the PV map. The PV field of the asymmetric response at $t = 24$ h (Fig. 6a) shows that PV is strongly localized near the eyewall. The PV field of the outer core region (Fig. 6b) shows that the magnitude of PV is significantly smaller (at least two orders of magnitude) in comparison to the eyewall region. Examining the PV field at this and other times (not shown) reveals that strong PV signals associated with vortex Rossby waves decay substantially not too far from where thermal perturbations are initially introduced, indicating the existence of a stagnation radius (Montgomery and Kallenbach 1997) and a

\[\text{Eqn. (2.11) of Nolan et al. (2007) should have an overbar over } \theta_n.\]
stagnation height (Möller and Montgomery 2000) for vortex Rossby waves. The waves propagating beyond the stagnation height and radius are therefore gravity waves. Fig. 7 shows the eddy angular momentum fluxes and their tendencies,

\[
\frac{\partial (r\overline{v}_r)}{\partial t} = \frac{r}{\overline{\rho} r^2} \frac{\partial}{\partial r} (\overline{\rho} r^2 \overline{u}_a v_a) - \frac{r}{\overline{\rho}} \frac{\partial}{\partial z} (\overline{\rho} w_a v_a),
\]

at \( t = 24 \) h. It clearly depicts that eddy angular momentum flux divergences of the asymmetric response are heavily localized near the eyewall. This indicates that radiating gravity waves do not significantly interact with the axisymmetric mean flow in the outer core region. However, eddy angular momentum fluxes are pointed away from the core region, suggesting that angular momentum is indeed transported away from the core by radiating gravity waves (mostly in the radial direction). Therefore, the amount of angular momentum transported away from the vortex by radiating gravity waves may be computed by examining the change of angular momentum in the core of the vortex, where the core is defined between the center of the vortex and the stagnation radius and height of vortex Rossby waves.

c. Symmetric response and computing the loss rate of angular momentum

To compute the full response to rotating convective asymmetric heat sources, it is necessary to compute how the mean vortex responds to the heat sources as well. Rotating asymmetries continuously interact with the mean vortex circulation through eddy flux divergences. In response, the vortex adjusts to a new balanced state by dispersing the source of imbalance through mechanisms such as symmetric gravity wave radiation and secondary circulations. The symmetric response of the MR vortex to the heat sources can be computed in 3DVPAS by using the flux divergences arising from the asymmetric response as the symmetric forcing. Fig. 8a shows the time evolution of KE and APE of the symmetric response.
Expressions for KE and APE of the symmetric response were also derived in sec. 2b of Nolan et al. (2007) as

\[ KE_0 = \int_{0}^{Z} \int_{0}^{R} \left( \frac{1}{2} \rho (u_0^2 + v_0^2 + w_0^2 + 2\nu v_0) \right) 2\pi r dr dz , \]  
\[ n = 0, \]  \[ \cdots \]  
\[ (17) \]

\[ APE_0 = \int_{0}^{Z} \int_{0}^{R} \frac{1}{2} \rho \left( \frac{g^2}{\theta_{\text{ref}}^2} \right) (\theta_0^2 + 2\theta_p \theta_0) 2\pi r dr dz . \]  
\[ n = 0, \]  \[ \cdots \]  
\[ (18) \]

In Eqs. (17) and (18), the subscript ref refers to the background or reference environment and \( \theta_p = \theta - \theta_{\text{ref}} \) is the perturbation potential temperature (the warm core) that holds the vortex in hydrostatic and gradient wind balance. Symmetric KE and APE are defined as the difference between the basic state and the total flow (the basic state plus the symmetric perturbations). It seems from Fig. 8a that both symmetric KE and APE increase at a linear rate after \( t = 8 \) h. Unlike the asymmetric response, the symmetric KE and APE do not reach a steady state. This is because they measure KE and APE of the accumulated symmetric change induced by the symmetric adjustment processes due to the tendency from the asymmetric response, which in a nearly steady state continuously interacts with the basic state. The linear increase of the symmetric KE is mostly due to the primary circulation, as the symmetric KE associated with the secondary circulation reaches a nearly steady state after \( t = 10 \) h (not shown). Fig. 8b shows the 24-hour change in angular momentum \( (rv_0 \) at \( t = 24 \) h) caused by the symmetric response, and it is again heavily localized near the eyewall.

To compute the change of angular momentum in the core of the vortex, it is necessary to know the stagnation radius and height of vortex Rossby waves. Analytical expressions exist for baroclinic perturbations on an axisymmetric, barotropic monopole vortex in hydrostatic and gradient wind balance (Montgomery and Kallenbach 1997; Möller and Montgomery 2000;
Reasor et al. 2000). However, there is no analytical expression for baroclinic perturbations on an axisymmetric, baroclinic vortex, and the SC vortex is not a monopole (see Fig. 9b). Instead, we approximate the stagnation radius and height by carefully examining the evolution of PV field. In all cases we find there is no vortex Rossby wave propagation beyond a radius of twice the RMW nor above a height of 12 km. Therefore, for all simulations in this study the stagnation radius and height are set to be $r_{stag} = 2 \times $ RMW (58 km) and $z_{stag} = 12$ km. These scales are in agreement with observations (e.g., Tuttle and Gall 1995) and numerical simulations (e.g., Chen and Yau 2001; Wang 2002a, 2002b) of vortex Rossby waves in tropical cyclones. Although it may appear from Fig. 8b that the net angular momentum change in the core region could be positive, it is actually negative.

To compare our results to the CC03 angular momentum transport calculation, it is necessary to characterize the angular momentum transport as a loss rate. Fig. 8c shows the time evolution of the loss of angular momentum in the core of the MR vortex. The loss of angular momentum in the core region of the vortex is defined as

$$\frac{\Delta L}{L} = \frac{\int_{0}^{r_{stag}} \int_{0}^{z_{stag}} \rho v_o r dr dz}{\int_{0}^{z_{stag}} \int_{0}^{r_{stag}} \rho v r dr dz}. \quad (19)$$

After a short initial adjustment period, radiating gravity waves transport angular momentum continuously away from the MR vortex core at a nearly constant rate. Taking the difference between $t = 23$ h and $t = 24$ h, the loss rate of angular momentum in the core is 0.012% per hour, which is significantly smaller than the prediction of about 10% per hour by CC03.

5. Sensitivity Tests
a. Radial structure of tangential wind field

The control case simulation is repeated but with different radial structures of the tangential wind. One way is to use different radial profiles but with the same $v_{max}$ at the same RMW. This can easily be done by using RWS, GS, and SC vortices (Figs. 9a and b) as the basic state. Another approach is to fix the general form of radial profile but change only the rate at which tangential wind field decays outside the RMW. This can be done most easily by using the MR vortex but with different values of $a$ in Eq. (5). For that purpose, MR vortices with $a = 0.4$ and 0.6 are considered and their surface wind fields are shown in Fig. 9c.

Calculations (Table 2) show that there is a factor of 4 difference in the loss rate among the different wind profiles. This seems to be related to the tangential wind field outside the RMW. Since the tangential wind fields outside the RMW of the MR and SC vortices decay much more slowly than those of the RWS and GS vortices (Fig. 9a), MR and SC vortices are more inertially stable and “stiff” in a sense that it is harder for gravity waves to propagate radially away from the core. The same line of reasoning would suggest that decreasing the value of $a$ in the MR vortex leads to a smaller loss rate of angular momentum. This is supported by the results for $a = 0.4$ and 0.6 in Table 2. Regardless of different radial structures of the tangential wind field, the loss rates for all simulations are significantly smaller than that of CC03.

b. Environmental stratification

The effects of different environmental stratification are examined. A far field environment with a constant Brunt-Vaisala frequency $N_i^2$ may be defined by the following temperature profile,

$$T(z) = T_0 \exp\left(\frac{zN_i^2}{g}\right) + \left(\frac{g^2}{c_p N_i^2}\right) \times \left(1 - \exp\left(\frac{zN_i^2}{g}\right)\right),$$

(20)
where $N_i^2$ is the desired constant, and $T_0 = 300$ K is used. The pressure and temperature fields that hold the vortex in hydrostatic and gradient wind balance are computed as before (see sec. 3b). The MR vortex with $v_{max} = 40$ ms$^{-1}$ and the $n = 2$ asymmetries with $v_{rot} = 20$ ms$^{-1}$ (50 %) are used. Calculations (Table 3) show that increasing the stratification frequency leads to a smaller amount of angular momentum transported away by gravity waves. At the same time, however, the total amount of the APE injected during the 24-hour integration decreases with increasing stratification frequency. This is because for a given diabatic heating rate, the rate at which the APE is injected is inversely proportional to the stratification frequency, as shown by the APE exchange relation derived in sec. 2b of Nolan et al (2007):

$$\frac{dAPE}{dt}\bigg|_z = \int_0^R \int_0^2 \frac{\bar{\rho}}{\bar{\theta}^2} \frac{g^2}{N_i^2} \left( -\frac{\partial \bar{\theta}}{\partial r} u_n \bar{\theta}_n - \frac{\partial \bar{\theta}}{\partial z} \bar{w}_n \bar{\theta}_n + \bar{\theta}_n F_{\phi} + \bar{\theta}_n \bar{\theta}'_n \right) 2\pi r dr dz . \quad (21)$$

The $F$ term in Eq. (21) is frictional force on potential temperature (see sec. 2a of Nolan and Montgomery (2002) for the descriptions of all frictional forces). Because the total amount of APE injected is different, it is difficult to isolate the effects of changing the ambient stratification frequency without further tests. We examine another case with $N_i^2 = 2.0 \times 10^{-4}$ s$^{-2}$, but the heating rate is adjusted to result in an equal amount of injected APE ($5.65 \times 10^{14}$ J) as in the $N_i^2 = 1.0 \times 10^{-4}$ s$^{-2}$ case. This results in an estimated angular momentum loss rate of 0.0745 % h$^{-1}$, as compared to 0.0443 % h$^{-1}$ for the $N_i^2 = 1.0 \times 10^{-4}$ s$^{-2}$ case. Since group velocity is proportional to the stratification frequency, this result indicates that given the equal total amount of injected APE, increasing group velocity leads to higher angular momentum loss rate.

c. Strength and vertical structure of the basic state vortices

MR vortices with $v_{max} = 30$ ms$^{-1}$ (MR30) and $v_{max} = 50$ ms$^{-1}$ (MR50) are considered. In comparison to the control MR vortex, MR30 and MR50 vortices are vertically shallower and
deeper, respectively. These changes in depth reflect the different intensities of the tropical cyclones they represent. Asymmetries are again prescribed to rotate at 50% of the local tangential mean flow at $r = \text{RMW}$, so $v_{rot} = 15 \text{ ms}^{-1}$ for MR30 vortex and $v_{rot} = 25 \text{ ms}^{-1}$ for MR50 vortex. Calculations (Table 2) show that gravity waves become more efficient in transporting angular momentum away from the core as the vortex gets stronger, and this seems to be related with the strength of the warm core. Because stronger tangential wind field leads to stronger warm core structure, the upper level static stability is lower for MR50 vortex than MR30 vortex, especially at the region of large heating rate (approximately $20 \text{ km} < r < 40 \text{ km}$, $2.5 \text{ km} < z < 7.5 \text{ km}$; see Fig. 3b). This leads to a higher total amount of APE injected for the MR50 vortex case. As shown in the previous section (sec. 5b), this is because the rate at which the APE is injected is inversely proportional to the stratification frequency. The response to the imposed convective asymmetric heat sources (Figs. 10a and b) is more vigorous, and stronger gravity waves are generated (Figs. 10c and d), transporting more angular momentum away from the core. Now we examine another case with the MR30 vortex in which the heating rate is adjusted to result in the equal amount of total APE injected ($3.49 \times 10^{14} \text{ J}$) as in the MR50 vortex case. This results in the loss rate of $0.0579 \text{ % h}^{-1}$, as compared to $0.0215 \text{ % h}^{-1}$ for the MR50 vortex case. Given the equal amount of total APE injected, the angular momentum loss rate decreases as the vortex becomes stronger.

\textit{d. Azimuthal wavenumber and rotation speed of the convective heat sources}

First, the azimuthal wavenumber structure of the rotating heat sources is varied to $n = 3$ and 4. Since the rotation speed is closely related with azimuthal wavenumber as shown in Eq. (10), $v_{rot}$ is adjusted to 67% and 75% of the local tangential flow at $r = \text{RMW}$ for $n = 3$ and 4.
cases, respectively. The basic state is the MR vortex used in the control case. Calculations (Table 2) show that having a higher wavenumber structure leads to smaller angular momentum transport by gravity waves. This is because the response to higher wavenumber heat sources is weaker (Fig. 11), leading to weaker gravity wave radiation and thus smaller angular momentum transport.

Now the effect of varying rotation speed independent of azimuthal wavenumber is considered by examining the response to the heat sources rotating at \( r = \text{RMW} \) but with different speeds (between 25% and 75% of the local tangential flow with the increment of 5%). All other parameters are the same as in the control case. Fig. 12a summarizes the calculations of all cases considered, and it shows that the loss rate is largest when the heat sources are rotating at the 40% for \( n = 2 \), 45% for \( n = 3 \), and 50% for \( n = 4 \) cases. It also indicates that the relative rotation speed at which the loss rate is maximized increases as \( n \) increases, qualitatively similar to the way the rotation speed of the edge vortex Rossby waves increases as \( n \) increases according to Eq. (10).

Fig. 12b shows the calculations for the \( n = 2 \) case but with different values of the decay parameter \( a \) in (5). As \( a \) increases toward unity, the peak loss rate increases as well. This will be explained shortly.

Why are the loss rates largest at those particular rotation speeds? The answer seems to be related to how closely the response to the heat sources stays in phase with the actual rotating heat sources. Fig. 13 shows the horizontal cross section of potential temperature of the asymmetric response and potential temperature tendency of the rotating heat sources at \( t = 24 \text{ h} \) for \( n = 2 \) and \( v_{rot} = 30\%, \ 40\%, \text{ and } 50\% \) of the local tangential flow cases, respectively. Careful examination of Fig. 13 clearly indicates that the asymmetric response of the \( v_{rot} = 40\% \) case stays more in phase with the rotating heat sources. This means that the rotating heat sources of equal magnitude
inject more APE for $v_{rot} = 40\%$, because the rate at which APE is injected is proportional to $\overline{\theta_t \theta_r}$. This leads to the more vigorous response and stronger gravity wave generation (Fig. 14), which ultimately results in more angular momentum transport away from the core.

Stability calculations have shown that the MR vortex and all other vortices considered in this study are stable so that no exponentially growing modes are expected. However, there is a sharp vorticity gradient just outside the RMW that can support modes similar to these waves on the edge of a Rankine vortex. To identify these waves and their true phase speeds, we repeat the control case simulation, but the heat sources are turned off at $t = 12 \text{ h}$, the time at which the control simulation settles into a nearly steady state (see Fig. 4). In the first few subsequent hours, KE of the asymmetric response decreases (solid line in Fig. 15a) and the adjustment processes continue to generate gravity waves that propagate away from the source location (RMW). At about $t = 16 \text{ h}$ and afterwards, the only remaining waves near $r = \text{RMW}$ must be slowest decaying edge waves. To estimate their phase speed, PV at $x = \text{RMW}$, $y = 0 \text{ m}$, and $z = 6.75 \text{ km}$ is plotted after $t = 16 \text{ h}$ in Fig. 15b, clearly capturing the oscillations in PV associated with these rotating edge waves; $z = 6.75 \text{ km}$ is the altitude of the maximum PV response. Using the period of the last two oscillations (about 187 minutes) gives the angular velocity of the edge waves at $5.58 \times 10^{-4} \text{ s}^{-1}$, which is very close to the angular velocity associated with the asymmetric heat sources rotating at 40% ($0.4 \times 40 \text{ ms}^{-1} / \text{RMW} \approx 5.50 \times 10^{-4} \text{ s}^{-1}$) of the local mean tangential flow at the surface. The angular velocities associated with the heat sources rotating at 30% and 50% of the local mean tangential circulation are $4.12 \times 10^{-4} \text{ s}^{-1}$ and $6.88 \times 10^{-4} \text{ s}^{-1}$. From this perspective, the rotation speed at which the loss rate is largest is close to the phase speed of the slowest decaying edge waves that are dictated by the structure of the mean vortices. At this rotation
speed, the excitation of these waves leads to a more vigorous response, generating stronger gravity waves thus transporting more angular momentum away from the vortices.

Fig. 12b suggests that the peak loss rate of angular momentum increases as $a$ increases toward unity. To better understand this result, we repeat the control case simulation with the heating turned off at $t = 12$ h, but with $a = 0.4$ and $0.6$. Fig. 15a shows that the asymmetric KE decreases in both cases at a different rate. Fig. 15c focuses on the temporal evolution between $t = 12$ h and $14$ h of the asymmetric KE normalized by its value at $t = 12$ h. It shows that as $a$ increases toward unity the asymmetric KE decreases more slowly. Increasing the value of $a$ in the MR vortex moves the critical radius, where the angular velocity of the mean vortex is equal to the phase velocity of the wave, radially inward. At the same time, the radial PV gradient at the critical radius becomes less negative. Since the damping of the quasimodal edge waves is proportional to the magnitude the negative radial PV gradient (Schecter and Montgomery 2003), increasing $a$ toward unity makes the quasimode become closer to a neutral eigenmode (Schecter and Montgomery 2004; Hodyss and Nolan 2008). This would lead to a higher angular momentum loss rate, as shown in Fig. 12b.

e. An attempt to reproduce the calculation by CC03

The shallow water calculation of CC03 is similar to the calculation of the loss rate of angular momentum for barotropic perturbations on a barotropic, axisymmetric vortex. Here we repeat the control case simulation but with rotating heat sources having deeper, more barotropic vertical structure (MRBT case) by using $z_h = 8$ km and $\sigma_z = 7$ km in Eq. (9). The amplitude of this quasi-barotropic heat source is chosen to have an equal amount of vertically integrated diabatic heating as the heat source shown in Fig. 3. In addition, the response of a completely
barotropic version of the MR vortex of $v_{\text{max}} = 40 \text{ ms}^{-1}$ to the same “barotropic” rotating heat sources (BTMR case) is also computed. Calculations (Table 2) show that the loss rates of angular momentum remain much smaller than the estimate by CC03.

6. Discussion

Sensitivity tests have confirmed the control case result that gravity waves propagating away from the core of tropical-cyclone-like vortex do transport angular momentum, but the loss rate of angular momentum due to their transport is several orders of magnitude smaller than that estimated by CC03. To understand this significantly large difference, it is necessary to note the differences in the framework on which each calculation is based. CCL02 and CC03 modeled radiating gravity waves in shallow water equations in which the only allowed wave solutions of gravity waves are external, although they did use parameters that are representative of the internal waves observed in their numerically simulated tropical cyclone. In reality, gravity waves propagating away from tropical cyclones are internal waves in a stratified atmosphere, which are better represented in 3DVPAS.

In addition, Schecter (2008) recently examined the magnitude of angular momentum transport by radiating gravity waves in the parameter regime where both Rossby and Froude numbers are larger than unity. Calculations for a strongly perturbed category-5 barotropic vortex showed that the loss rate is on the order of 0.1% per each rotation period, whereas it was 10 % per each rotation period for CC03. Schecter (2008) suggested that the discrepancy is due to an incorrect assumption in CCL02 and CC03 of the Froude number being less than unity. The derivation in CC03 also seems to use a formula (Eq. (6) of CC03) for Lighthill radiation that applies only to an elliptical ($m = 2$) perturbation, although higher order deformations ($m \geq 3$) of
the vortex produce weaker radiation, at least when the Rossby number is much greater than unity and the Froude number is less than unity (D. A. Schecter 2009; personal communication). The Froude number for this study is larger than unity \( (v/(g'H)^{1/2} = 40\text{ms}^{-1}/(0.01 \times 9.81\text{ms}^{-2} \times 12000\text{m})^{1/2}\approx 1.2)\), where 0.01 is used to estimate the reduced gravity \( g' \). In fact, CC03 actually mentioned the discrepancy of extending the low Froude number approximation to the hurricane regime where the Froude number is larger than unity. However, they did not address the effect of the discrepancy.

Eq. (2) as derived by CC03 seems to contradict the results of Ford (1994). Ford showed under the small Froude number assumption that the rate at which the PV patch in shallow water equations is deformed decreases with increasing azimuthal wavenumber as shown in Eq. (44) of Ford. However, Eq. (2) by CC03 has the loss rate increasing with increasing azimuthal wavenumber. Our results are in qualitative agreement with Ford (1994).

7. Summary

The response of the balanced vortices modeled after tropical cyclones to purely asymmetric convective rotating heat sources has been investigated to examine whether gravity waves can transport a significant amount of angular momentum away from tropical cyclones, as suggested by CCL02 and CC03. A stagnation-location argument was applied to separate the dynamical impacts of vortex Rossby waves and gravity waves. Calculations and sensitivity tests confirm that gravity waves do transport angular momentum away from tropical cyclones. However, the loss rate of angular momentum due to radiating gravity waves is several orders of magnitude smaller than the estimate by CC03. The difference may be due to an incorrect
extension of the low Froude number approximation to the tropical cyclone regime and to the
differences between two-dimensional and three-dimensional adjustment processes.

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Table 1: The value of each parameter used to create the basic state MR, RWS, GS, and SC basic state vortices.

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<th>Vortex type</th>
<th>Parameter</th>
<th>Value</th>
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<td></td>
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<td></td>
<td>$\zeta_2$</td>
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<td>$b$</td>
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<td>$R_2 - R_1$</td>
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Table 2: Parameters that control the configuration of each simulation and computed loss rate of angular momentum. “MRBT” refers to the simulation that uses the MR vortex with the “barotropic” heat sources. “BTMR” refers to the simulation that uses the barotropic version of the MR vortex with the “barotropic” heat sources. See sec. 5e for the descriptions of the MRBT and BTMR cases. ‘*’ means the same as the control case (first row), and ‘X’ means that the parameter does not apply.

<table>
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<tr>
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<th>Lz (km)</th>
<th>$\alpha$</th>
<th>n</th>
<th>$v_{\text{rot}}$ (ms$^{-1}$)</th>
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<td>*</td>
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<td>1.75</td>
<td>*</td>
<td>25.0 (50%)</td>
<td>0.0215</td>
</tr>
<tr>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>3</td>
<td>26.8 (67%)</td>
<td>0.0032</td>
</tr>
<tr>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>4</td>
<td>30.0 (75%)</td>
<td>0.0024</td>
</tr>
<tr>
<td>MRBT</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>0.0016</td>
</tr>
<tr>
<td>BTMR</td>
<td>*</td>
<td>*</td>
<td>X</td>
<td>X</td>
<td>*</td>
<td>*</td>
<td>0.00002</td>
</tr>
</tbody>
</table>
Table 3: Angular momentum loss rates to different values of the environmental stratification frequency as discussed in sec. 5b.

<table>
<thead>
<tr>
<th>$N^2 \text{ (s}^{-2}\text{)}$</th>
<th>Total amount of APE injected (J)</th>
<th>Loss rate (% hr$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.8 \times 10^{-4}$</td>
<td>$1.05 \times 10^{15}$</td>
<td>0.1065</td>
</tr>
<tr>
<td>$0.9 \times 10^{-4}$</td>
<td>$8.11 \times 10^{14}$</td>
<td>0.0680</td>
</tr>
<tr>
<td>$1.0 \times 10^{-4}$</td>
<td>$5.65 \times 10^{14}$</td>
<td>0.0443</td>
</tr>
<tr>
<td>$1.1 \times 10^{-4}$</td>
<td>$3.90 \times 10^{14}$</td>
<td>0.0316</td>
</tr>
<tr>
<td>$1.5 \times 10^{-4}$</td>
<td>$1.06 \times 10^{14}$</td>
<td>0.0103</td>
</tr>
<tr>
<td>$2.0 \times 10^{-4}$</td>
<td>$2.98 \times 10^{13}$</td>
<td>0.0039</td>
</tr>
<tr>
<td>$5.0 \times 10^{-4}$</td>
<td>$1.31 \times 10^{11}$</td>
<td>0.0002</td>
</tr>
</tbody>
</table>