Available online at www.sciencedirect.com

SCIENCE DIRECT®

Physics Letters A 327 (2004) 9–14

PHYSICS LETTERS A

www.elsevier.com/locate/pla

Rotational dynamics of turbulence and Tsallis statistics

A.M. Reynolds [a,*], M. Veneziani [b]

[a] *Silsoe Research Institute, Wrest Park, Silsoe, Bedford MK45 4HS, UK*
[b] *RSMAS/MPO, University of Miami, Miami, FL, USA*

Received 11 March 2004; received in revised form 26 April 2004; accepted 28 April 2004

Available online 11 May 2004

Communicated by C.R. Doering

Abstract

The Langevin equation and its adaptations can reproduce accurately key aspects of fluid-particle motions in turbulent flows including: ballistic and normal-diffusive transport regimes, Lagrangian velocity autocorrelation functions and integral Lagrangian timescales. More recently it has become apparent that trajectory-rotations are also important dynamical quantities and that non-zero mean rotations are associated with suppressed rates of turbulent dispersion and oscillatory Lagrangian velocity autocorrelation functions. Here, it is shown that rotations of the Lagrangian velocity vector produced by the simplest and most widely used of such models coincide closely with the intense rotations measured in the recent seminal experiment by Zeff et al. [Nature 421 (2003) 146] and are described precisely by Tsallis statistics. Model predictions (Tsallis distributions) for the rotational statistics of the North Atlantic Ocean (region between 35–43 °W and between 36–42 °N) are found to be in close agreement with simulation data produced by the Miami Isopycnic-Coordinate Ocean Model.

PACS: 02.50.Ga; 02.50.Ey; 05.10.Gg; 05.20.Jj; 47.27.Eq; 47.32.Cc

Keywords: Turbulence; Trajectory rotation; Tsallis statistics; Nonextensivity

Lagrangian properties of fluid motion are of fundamental importance to the understanding and prediction of transport and mixing in turbulent flows. In the large Reynolds number limit the acceleration autocorrelation function approaches a delta-function at the origin, corresponding to an uncorrelated (white noise) component in the acceleration and hence to a Markov process [1]. Lagrangian velocities in large Reynolds number turbulence can therefore be represented as a continuous Markovian process. At finite Reynolds number a better approximation is attained by replacing the white noise with noise that is temporally correlated over the Kolmogorov dissipative timescale. This is equivalent to modelling Lagrangian accelerations as a continuous Markovian process [1]. The approach is known to provide a good description of the dissipation subrange structure of Lagrangian velocity and acceleration statistics. The simplest Markovian model for the evolution of Lagrangian accelerations in a turbulent flow is prescribed by

---

[*] Corresponding author.
E-mail address: andy.reynolds@bbsrc.ac.uk (A.M. Reynolds).

0375-9601/$ – see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.physleta.2004.04.078

$$\begin{aligned} \mathrm{d}A_i = {} & -\big(T^{-1}+t_\eta^{-1}\big)(A_i-\varepsilon_{ijk}\Omega_j u_k)\,\mathrm{d}t \\ & - T^{-1}t_\eta^{-1}u_i\,\mathrm{d}t + \varepsilon_{ijk}\Omega_j u_k\,\mathrm{d}t \\ & + \sqrt{2\sigma_u^2\big(T^{-1}+t_\eta^{-1}\big)T^{-1}t_\eta^{-1}}\,\mathrm{d}\xi_i, \end{aligned} \tag{1}$$

where u is the velocity of a fluid-particle along its trajectory at time t, σ_u^2 is velocity variance and $\mathrm{d}\xi_i$ are increments of independent Wiener processes and have mean zero and variance $\mathrm{d}t$ [2]. The model timescales, T and t_η, can be associated with the 'energy-containing' (i.e., the larger-scales of motion that make the dominant contribution to velocity) and the 'dissipative' scales of motion, and are determined uniquely by the specification of the Lagrangian integral timescale, $T_\mathrm{L} = T + t_\eta$, and the *conditional* acceleration variance, $\sigma_A^2 = \sigma_u^2/Tt_\eta$. By construction the dissipative sub-range form of the Lagrangian velocity structure function is compatible with Kolmogorov similarity scaling and the distribution of simulated velocities and conditional accelerations is jointly Gaussian,

$$\begin{aligned} P(A,u) = {} & \big(2\pi\sigma_u^2\sigma_A^2\big)^{-3}\exp\left(-\frac{(A_i-\varepsilon_{ijk}\Omega_j u_k)^2}{2\sigma_A^2}\right) \\ & \times\exp\left(-\frac{u_i^2}{2\sigma_u^2}\right). \end{aligned} \tag{2}$$

The conditional mean-acceleration $\langle A_i|u\rangle = \varepsilon_{ijk}\Omega_j u_k$ (when non-zero) endows trajectories with a preferred sense of rotation. In the long-time limit ($t > T_\mathrm{L}$) or when $t_\eta/T \to 0$ the stochastic model (1) effectively reduces to the generalized Langevin equation,

$$\mathrm{d}u_i = -u_i T_\mathrm{L}^{-1}\,\mathrm{d}t + \varepsilon_{ijk}\Omega_j u_k\,\mathrm{d}t + \sqrt{2\sigma_u^2 T_\mathrm{L}^{-1}}\,\mathrm{d}\xi_i, \tag{3}$$

introduced by Borgas et al. [3] in their assessment of the impact of trajectory rotation upon turbulent dispersion. Non-zero mean trajectory rotation was found to be associated with: spiralling trajectories; oscillatory Lagrangian velocity autocorrelation functions; suppressed rates of turbulent dispersion for given turbulent kinetic energy and mean dissipation rate; and skew diffusion. Veneziani et al. [4] subsequently extracted estimates for the inverse rotational timescale, Ω, from historical records for acoustically-tracked sub-surface floats in the north-western Atlantic Ocean. They demonstrated that Eq. (3) together with such estimates for Ω can reproduce accurately the observed oscillatory Lagrangian velocity autocorrelation functions and the observed suppressed rates of dispersion of cyclones. In this Letter this analysis is extended by comparing model predictions for the distribution of incremental rotations with simulation data produced by the Miami Isopycnic-Coordinate Ocean Model (MICOM) [5]. MICOM is a high-resolution (1/12 degree grid spacing) general ocean circulation model configured with realistic topography and stratification (19 isopycnal layers in the vertical plus a surface mixed layer). The external forcing was based upon monthly atmospheric data for 1979–1999 supplied by the European Centre for Medium-range Weather Forecasts. It will be shown that rotations of the Lagrangian velocity vector produced by the stochastic models (1) and (3) are described precisely by Tsallis statistics and that these statistics coincide closely with the MICOM simulation data. That is, Tsallis statistics are derived from a concrete dynamical model that is physically well-founded rather than used to provide an empirical fit to probability distributions; a widely adopted [6–9] but much criticized approach [10,11]. Tsallis statistics with a prescribed 'q-value' have been established from concrete dynamical equations for only two other cases. Namely, in the dynamics described by the Langevin equation with a judicious choice of fluctuating parameters [12] and in the sensitivity to initial conditions, at the edge of chaos, of certain non-linear one-dimensional maps [13]. Note also, that the application of equilibrium statistics, such as Tsallis equilibria, to single degrees of freedom in dissipative systems described by Langevin equations is well founded unlike their application to a more complete description of turbulence [10].

The distribution of rotations, $S_i = \varepsilon_{ijk}u_j A_k$, of the Lagrangian velocity vector is readily calculated from (2):

$$\begin{aligned} P(S_i) \equiv {} & \big(2\pi\sigma_u^2\sigma_A^2\big)^{-3}\int\exp\left\{-\frac{A_i^2}{2\sigma_A^2}\right\}\exp\left\{-\frac{u_i^2}{2\sigma_u^2}\right\} \\ & \times\delta(S_i-\varepsilon_{ijk}u_j A_k)\,\mathrm{d}^3u\,\mathrm{d}^3A \\ = {} & N\exp\left\{-\frac{\left|\Omega\sigma_u \pm\sqrt{\sigma_A^2+\sigma_u^2\Omega^2}\right||S_i|}{\sigma_A^2\sigma_u}\right\}, \end{aligned} \tag{4}$$

where N is a normalization factor. The '+' sign applies when $S_i < 0$ and the '−' sign applies when $S_i \geqslant 0$. It follows directly from (4) that when $\Omega = 0$

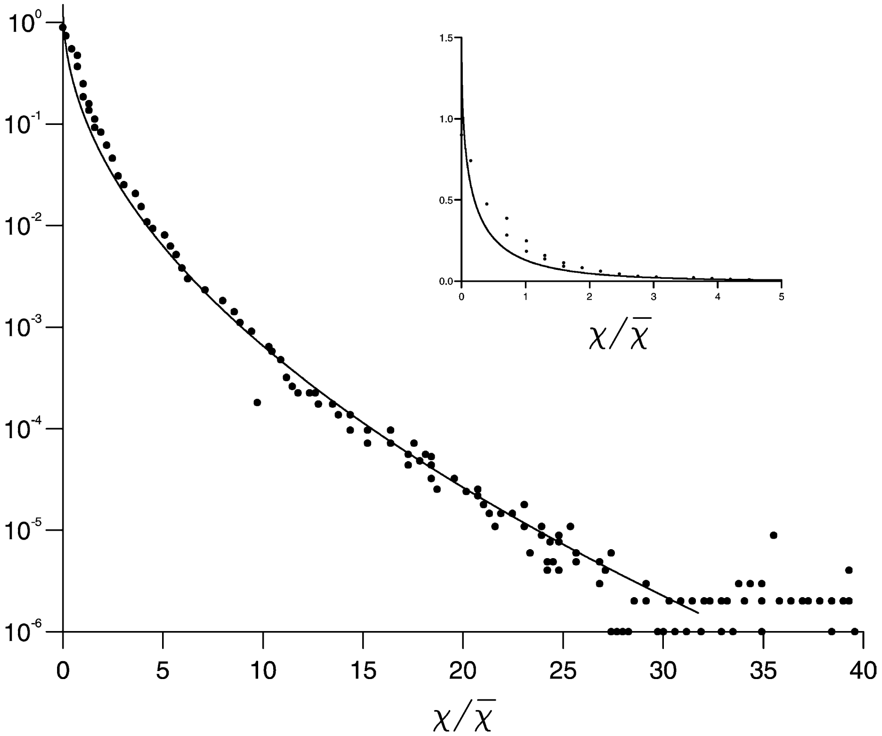

Fig. 1. Comparison of the predicted distribution of squared rotations, $\chi = S^2$ (solid-line), and the measured distribution of enstrophy (symbols) for a turbulent flow with Reynolds number $R_\lambda = 54$, based on the Taylor microscale.

the distribution of the Lagrangian analogue of enstrophy, $\chi = S^2$, is given by

$$P(\chi) = \frac{\exp(-\chi/\sigma_A\sigma_u)}{2\sigma_A^2\sigma_u^2}.$$

In contrast with the application of Tsallis statistics to description of Lagrangian acceleration statistics [6], fourth- and higher-order moments of $P(\chi)$ are convergent. Fig. 1 shows that except for the central part, $|\chi| < 2\bar{\chi}$ where $\bar{\chi}$ denotes the mean value of χ, this prediction coincides closely with the stretched exponential distribution, $P(\chi) \approx \exp(-\chi^{0.45\pm0.07})$ of enstrophy per se measured in recent seminal laboratory-scale experiments [14]. The predicted pseudo flatness statistic

$$F = \int_0^{\chi=30\bar{\chi}} \chi^4 P(\chi)\,\mathrm{d}\chi \left(\int_0^{\chi=30\bar{\chi}} \chi^2 P(\chi)\,\mathrm{d}\chi \right)^{-2} \approx 22$$

is coincident with the estimate of the pseudo flatness statistic obtained from the experimental data for $|\chi| \leqslant 30\bar{\chi}$. In obtaining this correspondence between model and experiment, the acceleration variance was taken to be prescribed by Kolmogorov similarity scaling and given by $\sigma_A^2 = a_0\varepsilon^{3/2}\nu^{-1/2}$ where the parameter $a_0 = 0.13R_\lambda^{0.64}$, $R_\lambda = (15/\varepsilon\nu)^{1/2}\sigma_u^2$ is the Reynolds number based upon the Taylor microscale, ε is the mean rate of dissipation of turbulent kinetic energy divided by fluid density and ν is the kinematic viscosity [1]. Parameter values were taken from experiment [14] and are given by $R_\lambda = 54$, $\varepsilon = 0.0336\ \mathrm{cm}^2\,\mathrm{s}^{-3}$ and $\nu = 0.01\ \mathrm{cm}^2\,\mathrm{s}^{-1}$. The close correspondence between model and experiment suggests that the intense turbulent rotations identified by Zeff et al. [14] are encapsulated within standard adaptations of the Langevin equation and do not constitute a new dynamical process.

Rotations such as S_3 define the inverse rotational timescale, $\omega_3 = S_3/(u_1^2 + u_2^2)\Delta t$, whose mean value is Ω_3. The conditional distribution $p(\omega_3|u)$ is Gaussian with mean $\langle\omega_3\rangle = \Omega_3$ and variance $\sigma_\omega^2 = \sigma_A^2/(u_1^2 + u_2^2) \equiv \beta^{-1}$. β is the sum of two independent

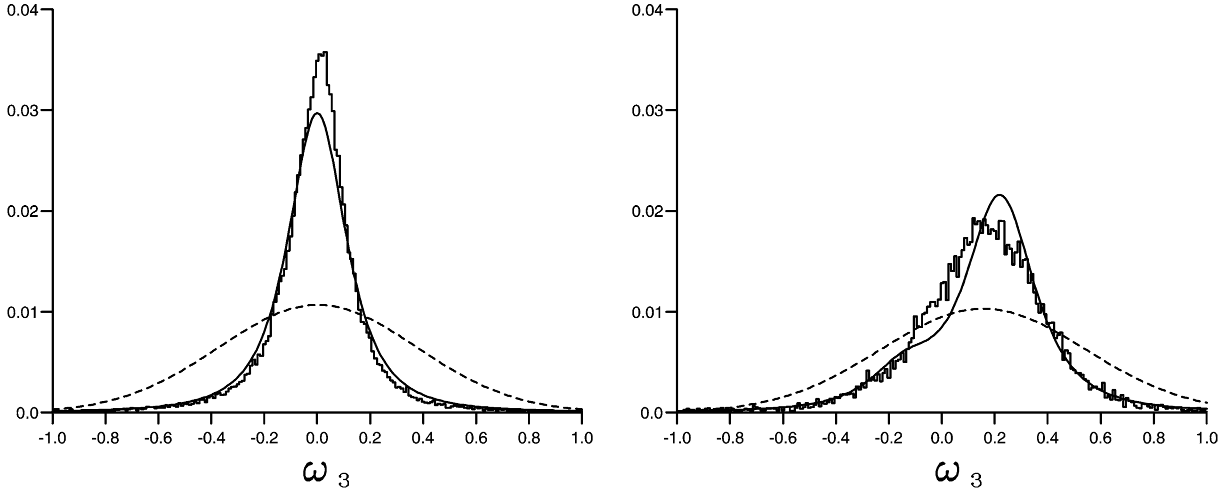

Fig. 2. Distributions of inverse rotational timescales for non-looping (left) and looping (right) trajectories extracted from the MICOM simulations (staircases) and produced by $q = 5/3$ Tsallis statistics (smooth solid-lines). Also shown for comparison are Gaussian distributions with equivalent means and variances (dashed-lines).

Gaussian random quantities and consequently is chi-squared distributed with degree $n = 2$. That is, the probability distribution of β is given by

$$f(\beta) = \frac{1}{\Gamma(n/2)}\left(\frac{n}{2\beta_0}\right)^{n/2}\beta^{n/2-1}\exp\left(-\frac{n\beta}{2\beta_0}\right), \quad (5)$$

where $\beta_0 = n\sigma_u^2/\sigma_A^2$. Therefore, the corresponding unconditional distribution of inverse rotational timescales,

$$\begin{aligned} p(\omega_3) &= \int f(\beta)p(\omega_3|\beta)\,\mathrm{d}\beta \\ &= \frac{\Gamma(n/2+1/2)}{\Gamma(n/2)}\left(\frac{\beta_0}{2\pi}\right)^{1/2} \\ &\quad \times \left[1 + \frac{\beta_0(\omega_3 - \Omega_3)^2}{2}\right]^{-n/2-1/2} \end{aligned} \quad (6)$$

is a $q = 1 + 2/(n + 1) = 5/3$ Tsallis distribution [12]. This value of q is consistent with a recent estimate, $q = 1.9 \pm 0.2$ obtained by fitting Tsallis distributions to vorticity distributions obtained from laboratory-scale experimental studies of rotating quasi-two-dimensional turbulence [8].

Determination of the unconditional distribution of inverse rotational timescales, ω_3, associated with incremental rotations $\Delta S_i = \varepsilon_{ijk}u_j\Delta u_k$ produced by the generalized Langevin equation (3) mirrors that outlined above for stochastic model (1). It is readily found that $P(\omega_3)$ is again a $q = 5/3$ Tsallis distribution but with $\beta_0 = n\sigma_u^2/\sigma_a^2$ defined in terms a pseudo acceleration variance $\sigma_a^2 = 2\sigma_u^2/T\Delta t$.

Analysis of the MICOM data revealed that 76.5% of the trajectories were 'non-looping' with $\Omega_3 \approx 0$ day^{-1}, $\sigma_u^2 \approx 40$ cm^2 s^{-2}, $T \approx 12$ days, 20% of the trajectories were 'cyclones' with $\Omega_3 = 0.22$ day^{-1}, $\sigma_u^2 \approx 110$ cm^2 s^{-2}, $T \approx 10$ days and that 3.5% were 'anticyclones' with $\Omega_3 = -0.15$ day^{-1}, $\sigma_u^2 \approx 70$ cm^2 s^{-2}, $T \approx 10$ days. In each case $t_\eta \approx 5$ days. Fig. 2 shows that the MICOM simulation data for the distribution of inverse rotational timescales for the non-looping trajectories is well represented by a $q = 5/3$ Tsallis distribution (6). It is also evident that the distribution of inverse rotational timescales for the 'looping' trajectories is well represented by the weighted sum of two $q = 5/3$ Tsallis distributions, i.e., by

$$p(\omega_3) = \frac{0.2p_c(\omega_3) + 0.035p_a(\omega_3)}{0.235},$$

where $p_c(\omega_3)$ and $p_a(\omega_3)$ are the $q = 5/3$ Tsallis distributions associated with cyclones and anticyclones. The discrepancy between peak values of the distributions extracted from the MICOM data and produced by the Tsallis can be reduced considerably when the timescales T are varied by amount, ±1–2 days, comparable with the uncertainty associated with their determination. The resultant narrowing of the Tsallis

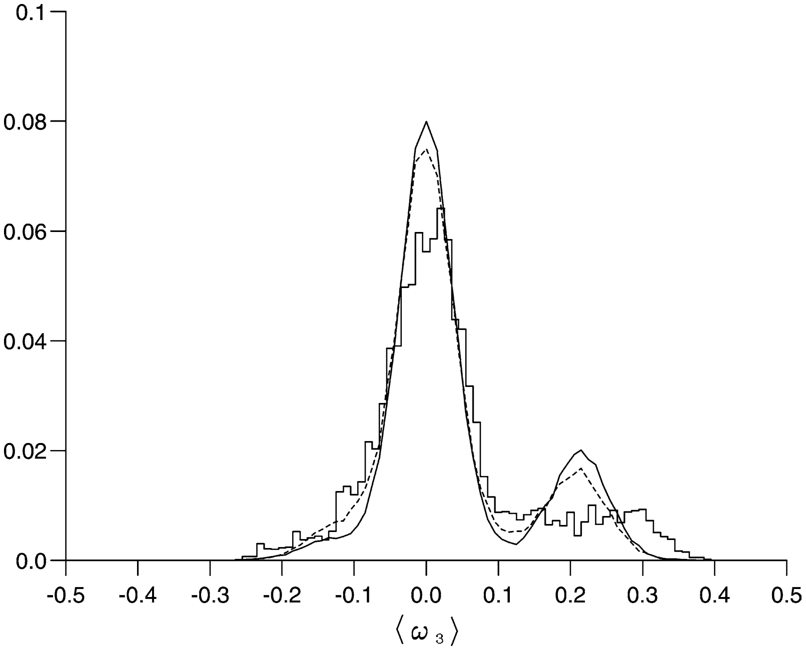

Fig. 3. Distributions of inverse rotational timescales produced by MICOM simulations (staircase), by the generalized Langevin equation (solid-line) and by a simple extension of the generalized Langevin equation incorporating transitions between looping and non-looping trajectories (dashed-line).

distributions does not affect significantly the correspondence with the MICOM data. This finding suggests that incremental rotations of both non-looping and looping trajectories in oceanic turbulence are described by Tsallis statistics.

Longer-time rotational statistics were extracted from MICOM simulations by taking 60-day running averages over single-trajectory records that were between 60 and 180 days in length. That is,

$$\langle \omega_3(t) \rangle = \frac{1}{120} \sum_{n=1}^{120} \omega(t + n\Delta t),$$

where $\Delta t = 0.5$ days. These running averages are fluctuating quantities that differ from the stationary mean value that may be obtained by averaging over much longer records. The statistics of $\langle \omega_3 \rangle$ can be described by (6) provided that the associated β-term,

$$\beta = \sigma_A^{-1} \sum_{i=1}^{120} u^2(t + i\Delta t) + v^2(t + i\Delta t)$$

is chi-squared distributed. This is very nearly the case because the integral timescale for $u^2 - \sigma_u^2$ is approximately 5 days which is much shorter than the 60 day averaging period. Consequently the β-term can be represented approximately as the sum of 24 independent squared-velocity terms. It is therefore anticipated that the long-time MICOM simulation data for $p(\langle \omega_3 \rangle)$ can be reproduced by 3 appropriately weighted Tsallis distributions with $q = 1 + 2/(24 + 1) \approx 1.08$, i.e., by

$$p(\langle \omega_3 \rangle) = 0.765 p_n(\langle \omega_3 \rangle) + 0.2 p_c(\langle \omega_3 \rangle) + 0.035 p_a(\langle \omega_3 \rangle),$$

where $p_n(\langle \omega_3 \rangle)$, $p_c(\langle \omega_3 \rangle)$ and $p_a(\langle \omega_3 \rangle)$ are the Tsallis distributions associated with the non-looping trajectories, with the cyclones and with the anticyclones. In the limit that $q \to 1$, Tsallis distributions reduce to Gaussian distributions [15].

Fig. 3 shows a comparison of the long-time MICOM simulation data and predictions produced by the generalized Langevin equation (3) and, by a simple extension of (3) accounting for the observed transitions between looping and non-looping trajectories. In accordance with the MICOM simulation data the probability, $P_{l \to n}$, of a transition from looping to non-looping behaviour within a hundred day period was taken to be 6.5%. The corresponding probability of a transition from non-looping to looping behaviour, $P_{n\to l} = \frac{23.5}{76.5} P_{l\to n}$ ensured compatibility with the MICOM simulation data for the average-populations of non-looping (76.5%) and looping (23.5%) trajectories. It is evident from Fig. 3 that the long-time MICOM simulation data is well predicted by the generalized Langevin equation and that transitions between looping and non-looping trajectories account for some of the discrepancies between the MICOM simulation data and the model predictions. The predicted secondary maxima in place of the observed plateau feature in $P(\langle\omega_3\rangle)$ for $\omega_3 > 0$ may be attributed to the existence of a broad, rather than δ-function, distribution of mean rotations, Ω_3, associated with the cyclones. As anticipated, the distribution of $\langle\omega_3\rangle$ produced by the generalized Langevin equation was found to be described almost precisely by the weighted sum of 3 $q = 1.08$ Tsallis distributions.

The results of this study suggest that rotations in oceanic flows and other turbulent flows are described by Tsallis statistics and can be modelled accurately by generalized forms of the Langevin equation.

Acknowledgements

We thank Z.D. Garraffo and E.P. Chassignet for providing the MICOM simulation data from which the Lagrangian statistics were determined. This work has benefited from many stimulating discussions with A. Griffa about oceanic turbulence.

References

[1] B.L. Sawford, Phys. Fluids A 3 (1991) 1577.
[2] A.M. Reynolds, Physica D 172 (2002) 124.
[3] M.S. Borgas, T.K. Flesch, B.L. Sawford, J. Fluid Mech. 279 (1997) 69.
[4] M. Veneziani, A. Griffa, A.M. Reynolds, A.J. Marino, J. Phys. Oceanogr., in press.
[5] http://oceanmodeling.rsmas.miami.edu/MICOM/.
[6] C. Beck, Phys. Lett. A 287 (2001) 240.
[7] C. Beck, G.S. Lewis, H.L. Swinney, Phys. Rev. E 63 (2001) 035303(R).
[8] S. Jung, B.D. Storey, J. Aubert, H.L. Swinney, Physica D, in press.
[9] K.E. Daniels, C. Beck, E. Bodenschatz, Physica D, in press.
[10] T. Gotoh, R.H. Kraichnan, Physica D, in press.
[11] D.H. Zanette, M.A. Montemurro, Phys. Lett. A, in press.
[12] C. Beck, Phys. Rev. Lett. 87 (2001) 180601.
[13] M.L. Lyra, C. Tsallis, Phys. Rev. Lett. 80 (1998) 53.
[14] B.W. Zeff, D.D. Lanterman, R. McAllister, R. Roy, E.J. Kostelich, D.P. Lathrop, Nature 421 (2003) 146.
[15] C. Tsallis, R.S. Mendes, A.R. Plastino, Physica A 261 (1998) 534.